Walrasian Equilibria in Indivisible and Divisible Settings: Programing Duality

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In this set of notes we review linear and convex programming and Lagrangian duality. After the review introduce models of good allocation in indivisible and divisible setting and define the Walrasian equilibrium in both contexts. We show that in both cases equilibrium allocations correspond to the optimal arguments of the primal of the equilibrium allocations, while the optimal variables of the corresponding dual correspond to the equilbirium prices.

1 Linear and Convex Programming Duality

1.1 Lagrangian Duality Theory

We provide the machinery given in this section without proof. If you are interested to understand some of the tools given in this section you can explore courses such as Numerical Optimization or even Differential Geometry. We present here a generalization of the Lagrangian Duality Theorem, called the Karush–Kuhn–Tucker theorem. These tools will then allow us to derive important facts about Linear and Convex programming.

Consider the optimization problem P of the following form called the **primal** problem:

$$\min_{\boldsymbol{x}} \quad f_0(\boldsymbol{x}) \tag{1}$$

$$\text{Constrained by} \quad f_i(\boldsymbol{x}) \leq 0 \quad \forall i \in \{1, \dots, m\} \tag{2}$$

$$\text{And} \quad f_i(\boldsymbol{x}) = 0 \quad \forall i \in \{m+1, \dots, p\} \tag{3}$$

where f_i's are any functions. We define the Lagrangian function, L, corresponding to the above optimization problem P as follows:

$$L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f_0(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i f_i(\boldsymbol{x}) + \sum_{i=m+1}^{p} \mu_i f_i(\boldsymbol{x}) \tag{4}$$

$$\tag{5}$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are called **slack variables**. These variables are called slack variables because by setting them wisely we obtain a function whose minimum corresponds exactly to that of the problem P. We now show how we should set these slack variables such as to obtain a function whose minima corresponds to the minima of the optimzation problem P.

Observe that for every feasible $\boldsymbol{x}$, $\lambda \geq \mathbf{0}$ and $\boldsymbol{\mu}$, $f_0(\boldsymbol{x})$ is bounded below by the Lagrangian, that is:

$$\forall \lambda \in \mathbb{R}^m_+, \mu \in \mathbb{R}^{p-m} \quad f_0(\boldsymbol{x}) \geq L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \tag{6}$$

The above inequality implies the following:

$$f_0(\boldsymbol{x}) \geq \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{\mu}} L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \tag{7}$$

Before we move further, let's look at the quantity below:

$$\max_{\boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{\mu}} L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{\mu}} f_0(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i f_i(\boldsymbol{x}) + \sum_{i=m+1}^{p} \mu_i f_i(\boldsymbol{x}) \tag{8}$$

$$= f_0(\boldsymbol{x}) + \max_{\boldsymbol{\lambda} \geq \mathbf{0}} \sum_{i=1}^{m} \lambda_i f_i(\boldsymbol{x}) + \max_{\boldsymbol{\mu}} \sum_{i=m+1}^{p} \mu_i f_i(\boldsymbol{x}) \tag{9}$$

$$= f_0(\boldsymbol{x}) + \sum_{i=1}^{m} \max_{\lambda_i \geq 0} \lambda_i f_i(\boldsymbol{x}) + \sum_{i=m+1}^{p} \max_{\mu_i} \mu_i f_i(\boldsymbol{x}) \tag{10}$$

Observe the following:

$$\max_{\lambda_i \geq 0} \lambda_i f_i(\boldsymbol{x}) = \begin{cases} 0 & \text{if } f_i(\boldsymbol{x}) \leq 0 \\ \infty & \text{Otherwise} \end{cases} \tag{11}$$

$$\max_{\mu_i} \mu_i f_i(\boldsymbol{x}) = \begin{cases} 0 & \text{if } f_i(\boldsymbol{x}) = 0 \\ \infty & \text{Otherwise} \end{cases} \tag{12}$$

That is, by taking the maximum over the slack variables $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ we essentially obtained a function where all feasible values $\boldsymbol{x}$ of the program P corresponds to the values of $f_0(\boldsymbol{x})$, and to infinity for all infeasible values. As a result, we have:

$$\min_{\substack{\forall i \in \{1,\ldots,m\} f_i(\boldsymbol{x}) \leq 0 \\ \forall i \in \{m+1,\ldots,p\} f_i(\boldsymbol{x}) = 0}} f_0(\boldsymbol{x}) = \min_{\boldsymbol{x}} \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{\mu}} L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \tag{13}$$

We now introduce one of the most important results in Operations and Research, the Karush–Kuhn–Tucker theorem. The Karush–Kuhn–Tucker theorem is extremely important as it allows us to transform a constrained optimization problem to an unconstrained one.

Theorem 1.1. *Karush–Kuhn–Tucker theorem*
Let $L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ be the Lagrangian function corresponding to the optimization problem P. If $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ is a saddle point of L with $\boldsymbol{\lambda}^* \geq \mathbf{0}$, then $\boldsymbol{x}^*$ is optimal for the program P. Suppose that for all $i = 1, \ldots, p$, $f_i(\boldsymbol{x})$'s are all convex and there exists a $\boldsymbol{x}$ such that $\forall i = 1, \ldots, m, \;\; f_i(\boldsymbol{x}) \leq 0$, then the optimal $\boldsymbol{x}$ has an associated $(\boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ such that $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ is a saddle point of $L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})$.

We now link the theorem we have just studied to another cool result in game theory. In 2 person zero-sum games, a Nash equilibrium is saddle-point of the payoff function. One important results from zero-sum games is the Minimax theorem:

Theorem 1.2. *Minimax Theorem Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be compact convex sets. If $f : X \times Y \to \mathbb{R}$ is a continuous function that is concave-convex, i.e. $f(\cdot, y) : X \to \mathbb{R}$ is concave for fixed y, and $f(x, \cdot) : Y \to \mathbb{R}$ is convex for fixed x Then we have that*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y)$$

The implication of the above theorem is: $\min_{\boldsymbol{x}} \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{\mu}} L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{\mu}} \min_{\boldsymbol{x}} L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})$, assuming that our constraints are all convex which makes the set of feasible inputs for the Lagrangian compact and convex (+ the Lagrangian can be shown to be convex concave), our original primal program can be then expressed as a different program where with variables coresponding to the slack variables. This is the intuition behind deriving another program called **the dual** from the lagrangian. The dual variables often have meaning and can be used to solve for an additional problem.

The Lagrange dual function of a program P with Lagrangian function L is defined as:

$$g(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \inf_{\boldsymbol{x}} L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \tag{14}$$

Now, consider the following optimization program called the **dual program** of the primal program:

$$\max_{\boldsymbol{\lambda}, \boldsymbol{\mu}} \quad g(\boldsymbol{\lambda}, \boldsymbol{\mu}) \tag{15}$$

$$\text{Constrained by} \quad \boldsymbol{\lambda} \geq \mathbf{0} \tag{16}$$

Note that the dual function g is concave, even when the initial problem is not convex, because it is a point-wise maximum of affine functions.

1.2 Linear Programming

Linear Programming is a mathematical method that allows to find the variables that maximize or minimize a function that is constrained by a set of linear constraints. That is, linear programming refers to the set of methods that allow us to solve problems defined by the inputs $(\boldsymbol{A}, \boldsymbol{b}.\boldsymbol{c})$ and that can be expressed in the following canonical form :

Primal:

$$\min_{\boldsymbol{x}} \quad \boldsymbol{b}^T \boldsymbol{x} \tag{17}$$

$$\text{Constrained by} \quad \boldsymbol{A}^T \boldsymbol{x} \geq \boldsymbol{c} \tag{18}$$

$$\text{And} \quad \boldsymbol{x} \geq \mathbf{0} \tag{19}$$

We will now derive the **dual** program of the problem above using the machinery introduced in the previous section. Since the above expression is the canonical form for any linear program, the dual that we derive will also give us a formula to easily find the dual of any linear program. Let's first calculate the Lagrangian of the above program:

$$L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \boldsymbol{b}^T\boldsymbol{x} + \boldsymbol{\lambda}^T(\boldsymbol{c} - \boldsymbol{A}^T\boldsymbol{x}) - \boldsymbol{\mu}^T\boldsymbol{x} \tag{20}$$
$$= \boldsymbol{x}^T\boldsymbol{b} + \boldsymbol{c}^T\boldsymbol{\lambda} - \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{\lambda} - \boldsymbol{x}^T\boldsymbol{\mu} \tag{21}$$
$$= \boldsymbol{c}^T\boldsymbol{\lambda} + \boldsymbol{x}^T(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{\lambda}) - \boldsymbol{x}^T\boldsymbol{\mu} \tag{22}$$

We can then calculate the dual objective function g:

$$g(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \inf_{\boldsymbol{x}}\{\boldsymbol{c}^T\boldsymbol{\lambda} - \boldsymbol{x}^T(\boldsymbol{A}\boldsymbol{\lambda} - \boldsymbol{b}) - \boldsymbol{\mu}^T\boldsymbol{x}\} \tag{23}$$

Notice that this function is exactly Lagrangian form of the following maximization problem:

$$\max_{\boldsymbol{\lambda}} \quad \boldsymbol{c}^T\boldsymbol{\lambda} \tag{24}$$
$$\text{Constrained by} \quad \boldsymbol{A}\boldsymbol{\lambda} \leq \boldsymbol{b} \tag{25}$$
$$\text{and} \quad \boldsymbol{\mu} \geq \mathbf{0} \tag{26}$$

Adding to this program the dual program feasibility constraint, we obtain **the dual of the standard form** of Linear programming where we renamed the slack variable $\boldsymbol{\lambda}$ as $\boldsymbol{y}$. The dual variables $\boldsymbol{y}$ are called **the dual variables**.

Dual:

$$\max_{\boldsymbol{y}} \quad \boldsymbol{c}^T\boldsymbol{y} \tag{27}$$
$$\text{Constrained by} \quad \boldsymbol{A}\boldsymbol{y} \leq \boldsymbol{b} \tag{28}$$
$$\text{And} \quad \boldsymbol{y} \geq \mathbf{0} \tag{29}$$

Note that different authors might refer to the minimization problem as the dual and the maximization problem as the primal.

The expressions $\boldsymbol{c}^T\boldsymbol{x}$ and $\boldsymbol{b}^T\boldsymbol{y}$ are respectively called the **objectives** of the primal and dual problems. Solutions $\boldsymbol{x}^*$ for the primal that satisfy the primal constraints $\boldsymbol{A}\boldsymbol{x}^* \leq \boldsymbol{b}, \quad \boldsymbol{x}^* \geq \mathbf{0}$ are called **feasible solutions**. Similarly, solutions $\boldsymbol{y}^*$ for the dual that satisfy the dual constraints $\boldsymbol{A}^T\boldsymbol{y}^* \leq \boldsymbol{c}, \quad \boldsymbol{y}^* \geq \mathbf{0}$ are called **feasible solutions**. $\boldsymbol{y}^*$. A linear program is said to be a **feasible program** iff there exists variables for the program that are feasible, otherwise the program is said to be an **infeasible program**.

A feasible variable $\boldsymbol{x}^*$ for the primal is called optimal iff $\forall \boldsymbol{x} \in \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \quad \boldsymbol{x} \geq \mathbf{0}\}, \quad \boldsymbol{c}^T\boldsymbol{x}^* \geq \boldsymbol{c}^T\boldsymbol{x}$. A linear program in the primal form is said to be bounded iff $\forall \boldsymbol{x} \in \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \quad \boldsymbol{x} \geq \mathbf{0}\}, \quad \boldsymbol{c}^T\boldsymbol{x} < \infty$.

A feasible variable $\boldsymbol{y}^*$ for the dual is called optimal iff $\forall \boldsymbol{y} \in \{\boldsymbol{y} \mid \boldsymbol{A}^T\boldsymbol{y} \leq \boldsymbol{c}, \quad \boldsymbol{y} \geq \mathbf{0}\}, \quad \boldsymbol{b}^T\boldsymbol{x}^* \geq \boldsymbol{b}^T\boldsymbol{x}$. A linear program in the primal form is said to be bounded iff $\forall \boldsymbol{y} \in \{\boldsymbol{y} \mid \boldsymbol{A}^T\boldsymbol{y} \leq \boldsymbol{c}, \quad \boldsymbol{y} \geq \mathbf{0}\}, \quad \boldsymbol{b}^T\boldsymbol{y} < \infty$

We do not discuss algorithms to solve linear programs in this presentation, however there are many polynomial time algorithms to solve linear programs such as the simplex or big M algorithms. These algorithms generally take as input $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ and return a tuple $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ which are respectively the optimal variables for the primal and dual problems.

Below are the two most important results from Linear Programming Duality that confirm our initial motivation of the dual using the minimax theorem:

Theorem 1.3. *Weak Programming Duality*
Let $\boldsymbol{x}$ be any feasible solution to the primal program P, and $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ be any feasible solution to the dual program of P. Then $f_0(\boldsymbol{x}) \geq g(\boldsymbol{\lambda}, \boldsymbol{\mu})$

Theorem 1.4. *Strong Programming Duality for Linear Programming*
Let $\boldsymbol{x}$ be any feasible solution to the primal of a linear program P, and $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ be any feasible solution to the dual program of P. Let $f_0(\boldsymbol{x})$ be the objective of the primal and let $g(\boldsymbol{\lambda}, \boldsymbol{\mu})$ be the objective of the dual. Let $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ be the optimal variables for the primal and dual respectively, then $f_0(\boldsymbol{x}^*) = g(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$.

1.3 Convex Programming

Convex Programming is a mathematical method that allows to find the variables that maximize or minimize a function that is constrained by a set of convex inequality constraints and affine equality constraints. That is, convex programming refers to the set of methods that allow us to solve problems defined by the inputs $(f_0, f_1, \ldots, f_m)$ and that can be expressed in the following canonical primal form:

Primal:

$$\min_{\boldsymbol{x}} \quad f_0(\boldsymbol{x}) \tag{30}$$

$$\text{Constrained by} \quad f_i(\boldsymbol{x}) \leq 0 \quad \forall i \in \{1, \ldots, m\} \tag{31}$$

$$\text{And} \quad h_i(\boldsymbol{x}) = 0 \quad \forall i \in \{m+1, \ldots, p\} \tag{32}$$

Note that different authors might refer to the minimization problem as the dual and the maximization problem as the primal.

It is harder to derive the dual in closed form like we did for linear programming, however this can be done by going through the lagrangian or using shortcuts with the help of Frenchel conjugates. More information about finding the dual of a convex primal program can be found in section 3 of [1].

Definition 1.5. ***Slater's condition*** *We say that the problem satisfies Slater's condition if it is strictly feasible, that is:*

$$\exists x_0 \in \mathcal{D} : f_i(x_0) < 0, \quad i = 1, \ldots, m, \quad h_i(x_0) = 0, \quad i = 1, \ldots, p$$

We can replace the above by a weak form of Slater's condition, where strict feasibility is not required whenever the function f_i is affine.

For our purposes, we present the following duality theorem to confirm our intuition from the minimax theorem that a dual exists with the same objective value.

Theorem 1.6. ***Strong duality via Slater condition***
If the primal problem (8.1) is convex, and satisfies the weak Slater's condition, then strong duality holds. That is, let $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ be the optimal variables for the primal and dual respectively, then $f_0(\boldsymbol{x}^*) = g(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$.

2 Combinatorial Auction Markets

Combinatorial Auctions are a type of smart market, in that they are built by researchers and engineers alike in order to solve complex allocation problems. Combinatorial auctions were first proposed by Rassenti, Smith, and

Bulfin (1982), for the allocation of airport landing slots [2]. In combinatorial auctions, agents can place bids on discrete/indivisible sets of items rather than just individual items (from where the name combinatorial, since bids are reported on a combination of items).

Combinatorial auction markets can be designed with different desiderata in mind. As a result, it is not possible to define a one size fits all "equilibrium outcome" condition. However, for our purposes we will consider the Walrasian equilibria of such markets and study the computational aspects of such markets.

2.1 Market Elements

A combinatorial auction market consists of:

1. Finite set of n heterogeneous **bidders** $[n]$
2. Finite set of m possibly interrelated **items/goods** $[m]$. WLOG, we assume that there is only one unit of a good in the market.
3. Each bidder $i \in [n]$ has
   - a **value function** $v_i : 2^{[m]} \to \mathbb{R}_+$ which takes as input a subset of the items or goods and outputs a value that describes the preference of the bidder for the subset of goods in **monetary terms**. That is the value function only describes the maximum amount that the bidder is willing to pay for the subset of goods.
   - a **utility function** $u_i : 2^{[m]} \to \mathbb{R}_+$ defined as $u_i(S; \boldsymbol{p}) = v_i(S) - \sum_{j \in S} p_j$. This function is parametrized by the prices of the goods $\boldsymbol{p}$, takes a subset of goods as input and returns the utility derived by the bundle.
4. Note that in this setting bidders have no budgets, they can spend as much as they want. The feasibility constraint in this setting is that bidders will never pay more than their value for a bundle.

2.2 Outcome of the Market

An **allocation** $\boldsymbol{X}$ is a map from the powerset of goods, $2^{[m]}$, to buyers, $[n]$, represented as a matrix, s.t. $x_{iS} \in \{0, 1\}$ denotes whether if a subset of the goods $S \subseteq [m]$ has been allocated to buyer $i \in [n]$. Goods are assigned **prices** $\boldsymbol{p} \in \mathbb{R}^m$.

An allocation is **feasible** iff:

1. no more than the total number of goods available are allocated, i.e., $\forall j \in [m], \sum_{S \in 2^{[m]} \, : \, j \in S} \sum_{i \in [n]} x_{iS} \in \{0, 1\}$
2. every buyer is allocated only one bundle/subset of the goods, i.e., $\forall i \in [n], \sum_{S \in 2^{[m]}} x_{iS} \in \{0, 1\}$

An allocation $\boldsymbol{X}^*$ is **welfare maximizing** iff $\forall S' \subseteq [m], \sum_{S \in 2^{[m]}} x^*_{iS} v_i(S) \geq v_i(S')$. A combinatorial market outcome $(\boldsymbol{X}, \boldsymbol{p})$ is welfare maximizing iff $\boldsymbol{X}$ is welfare maximizing and $\boldsymbol{p}$ is feasible.

An outcome is a tuple $(\boldsymbol{X}, \boldsymbol{p})$ consisting of allocation and prices for goods respectively. An outcome is feasible iff:

1. the allocation is feasible
2. the price of the bundle allocated to a buyer is less than or equal to the value of that bundle, i.e. $\forall i \in [n], \sum_{S \in 2^{[m]}} \sum_{j \in S} x_{iS} \, p_j \leq \sum_{S \in 2^{[m]}} x_{iS} v_i(S)$

2.3 Equilibrium

The **demand set** function (i.e., Marshallian demand) of a bidder i, $D_i : \mathbb{R}^m \to 2^{[m]}$, takes as input prices $\boldsymbol{p}$ and returns the utility maximizing set of goods at given prices such that $D_i(\boldsymbol{p}) = \arg\max_{S \subseteq [m]} u_i(S; \boldsymbol{p})$

A feasible outcome $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ is called a **Walrasian equilibrium** iff:

1. the allocation given by the outcome is equal to the demand set of bidder i at prices $\boldsymbol{p}^*$, i.e., $\forall i \in [n], \sum_{i \in [n]} x_{iS} S = D_i(\boldsymbol{p}^*)$[1]
2. goods that are not allocated are priced at 0, i.e., $\forall [m] \in [m], \sum_{S \subseteq [m] \,:\, j \in S} \sum_{i \in [n]} x_{iS} = 0 \Rightarrow p_j = 0$

2.4 Solving for Equilibria: Linear Programming Duality

It turns out that Walrasian equilibria of Combinatorial markets can be obtained through linear programming. We first present a welfare maximizing program that provides allocations.

$$\max_{\boldsymbol{X}} \quad \sum_{i \in [n]} \sum_{S \subseteq [m]} x_{iS} v_i(S) \qquad \text{(Welfare Maximization Objective)} \tag{33}$$

$$\forall j \in [m], \quad \sum_{S \subseteq [m] \,:\, j \in S} \sum_{i \in [n]} x_{iS} \in \{0, 1\} \qquad \text{(No Goods Over-Allocated)} \tag{34}$$

$$\forall i \in [n], \quad \sum_{S \in 2^{[m]}} x_{iS} \in \{0, 1\} \qquad \text{(One Bundle Allocated per Buyer)} \tag{35}$$

$$\forall i \in [n], S \in 2^{[m]} \quad x_{iS} \in \{0, 1\} \qquad \text{(Variable Constraint)} \tag{36}$$

We now propose a linear programming relaxation of the winner determination problem. The only difference between this relaxed setting and the winner determination problem as we previously defined it, is that bidders can be allocated fractions of a bundle. The feasibility constraints are still the same but are recast to conform to this divisibility relaxation.

Primal

$$\max_{\boldsymbol{X}} \quad \sum_{i \in [n]} \sum_{S \subseteq [m]} x_{iS} v_i(S) \tag{37}$$

$$\forall j \in [m], \quad \sum_{S \subseteq [m] \,:\, j \in S} \sum_{i \in [n]} x_{iS} \leq 1 \tag{38}$$

$$\forall i \in [n], \quad \sum_{S \in 2^{[m]}} x_{iS} \leq 1 \tag{39}$$

$$\forall i \in [n], S \in 2^{[m]} \quad x_{iS} \geq 0 \tag{40}$$

Note that although this is a linear program and linear programs are solvable in polynomial time, the variable matrix $\boldsymbol{X}$ is of exponential size ($n \times 2^m$). As a result, in order to solve this LP efficiently it is required to store efficiently or simplify the valuations of the buyers such that the matrix of variables becomes of polynomial size.

[1] $\sum_{i \in [n]} x_{iS} S$ is a slight abuse of notation since S is a set and we are multiplying it by a number. In this case you think of $x_{iS} S$ as an indicator function which is equal to S if x_{iS} is equal to 1 and $\emptyset$ otherwise.

The dual of this program is given by:

Dual

$$\min_{\boldsymbol{u},\boldsymbol{p}} \quad \sum_{i\in[n]} u_i + \sum_{j\in[m]} p_j \tag{41}$$

$$\forall i \in [n], S \subseteq [m] \quad u_i + \sum_{j\in S} p_j \geq v_i(S) \tag{42}$$

$$\forall i \in [n], \quad u_i \geq 0 \tag{43}$$

$$\forall j \in [m], \quad p_j \geq 0 \tag{44}$$

$$\tag{45}$$

Note: variables $\boldsymbol{u}$ correspond to the constraints given by equation (39) in the primal, while variables $\boldsymbol{p}$ correspond to constraints given by equation (38) in the primal.

The use of the letters $\boldsymbol{u}$ and $\boldsymbol{p}$ in the dual as variables are purposeful. Before we interpret these two programs we have just proposed, we introduce the concept of a Walrasian equilibrium which will help us interpret the programs.

Primal: The objective of the primal is to find an allocation of goods that maximizes utilitarian social welfare constrained by the feasibility constraints on the allocation. This approach works in finding the allocations corresponding to Walrasian equilibria because as we will discuss Walrasian equilibria in this setting are welfare maximizing.

Dual: The dual variables $\boldsymbol{u}$ can be interpretted as the utility achieved by the bidders, while the variables $\boldsymbol{p}$ can be interpretted as the prices that enforce the allocation given by the primal. Enforce in this context means that if we were to sell our goods by posting prices $\boldsymbol{p}$ instead of auctioning them, under the assumption that the bidders are rational and want to maximize their utility, the allocation of goods to buyers would be exactly the allocation given by the primal. The objective of the primal is also equal to utilitarian welfare at equilibrium, this makes sense the objective is exactly the definition of welfare i.e., revenue of the auctioneer + utility of buyers. The objective here is to minimize welfare rather than maximizing it, since the variable we have to compute are not the allocation but rather the values that make up this utilitarian social welfare. Constraint (42) in the dual corresponds exactly to the definition of utility in our combinatorial auction market model.

Theorem 2.1. ***First Welfare Theorem - Indivisible Items*** *Let $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ be a Walrasian Equilibrium, then the allocation $\boldsymbol{X}^*$ maximizes social welfare. Furthermore, $\boldsymbol{X}$ maximizes social welfare over all fractional allocations. That is, the welfare obtained by the indivisible equilibrium allocation will be higher than the welfare obtained by any divisible allocation.*

Proof In a Walrasian equilibrium, each bidder receives his demand. Therefore, for every bidder i and every bundle S, we have $v_i(S_i^*) - \sum_{j\in S_i^*} p_j^* \geq v_i(S) - \sum_{j\in S} p_j^*$. since the fractional solution is feasible to the LPR, we have that for every bidder i, $\sum_s X_{i,S}^* \leq 1$ (Constraint 11.5), and therefore

$$v_i(S_i^*) - \sum_{j\in S_i^*} p_j^* \geq \sum_{S\subseteq M} X_{i,S}^* \left(v_i(S) - \sum_{j\in S} p_j^* \right)$$

The theorem will follow from summing the above inequality over all bidders, and showing that $\sum_{i\in N}\sum_{j\in S_i^*} p_j^* \geq \sum_{i\in N, S\subseteq M} X_{i,S}^* \sum_{j\in S} p_j^*$. Indeed, the left-hand side equals $\sum_{j=1}^m p_j^*$ since $S_1^*, \ldots, S_n^*$ is an allocation and the prices of unallocated items in a Walrasian equilibrium are zero, and the right-hand side is at most $\sum_{j=1}^{m} p_j^*$, since the coefficient of every price p_j^* is at most 1.

Theorem 2.2. ***Second Welfare Theorem - Indivisible Item*** *If an integral solution exists to the primal of the linear program, then a Walrasian equilibrium whose allocation is the given solution also exists. That is, any Welfare Maximizing outcome can be represented as a Walrasian equilibrium .*

Proof: An optimal integral solution for LPR defines a feasible efficient allocation $S_1^*, \ldots, S_n^*$. Consider also an optimal solution $p_1^*, \ldots, p_n^*, u_1^*, \ldots, u_n^*$ to DLPR. We will show that $S_1^*, \ldots, S_n^*, p_1^*, \ldots, p_n^*$ is a Walrasian equilibrium. Complementary-slackness conditions are necessary and sufficient conditions for the optimality of solutions to the primal linear program and its dual. Because of the complementary-slackness conditions, for every player i for which $x_{i,S_i^*} > 0$ (i.e., $x_{i,S_i^*} = 1$), we have that Constraint (11.8) is binding for the optimal dual solution, i.e.,

$$u_i^* = v_i\left(S_i^*\right) - \sum_{j \in S_i^*} p_i^*$$

Note that in the indivisible setting we are not guaranteed the existence Walrasian equilibria below is a counter-example to confirm this claim:

Consider two buyers, Alice and Bob, and two items $\{a, b\}$. Alice has a value of 2 for every nonempty set of items, and Bob has a value of 3 for the whole bundle $\{a, b\}$, and 0 for any of the singletons. The optimal allocation will clearly allocate both items to Bob. Therefore, Alice must demand the empty set in any Walrasian equilibrium. Both prices will be at least 2; otherwise, Alice will demand a singleton. Hence, the price of the whole bundle will be at least 4, Bob will not demand this bundle, and consequently, no Walrasian equilibrium exists for these players.

3 Fisher Market Model

The Fisher market model is the first model of a market that proves that a competitive equilibrium exists in a market, that is, there exists prices and allocation of goods that maximize the utility of buyers and clears the market. In other words, the fisher market proves that in a market prices exist such that the supply is equal to demand.

3.1 Market Elements

The market consists of:

1. Finite set of n heterogeneous **buyers** $[n]$.
2. Finite set of m heterogeneous **good** types $[m]$.
3. Each buyer $i \in [n]$ is characterized by:
   (a) a budget $b_i \in \mathbb{R}$. We assume that budgets are normalized $\sum_{i=1}^{n} b_i = 1$.
   (b) a utility function, $u_i : \mathbb{R}^m \to \mathbb{R}_+$, giving the utility that buyer i derives from each bundle of goods.
4. WLOG, we assume that there is only one unit of each good $j \in [m]$ and that each good is demanded by at least one person. The results/solutions we provide in this section can be applied to the more general settings in which the number of copies of each good is different than 1 and there are goods that are not demanded.

3.2 Outcome of the Market

An **allocation** $\boldsymbol{X}$ is a map from goods to buyers, represented as a matrix, s.t. $x_{ij} \geq 0$ denotes the amount of good $j \in [m]$ allocated to buyer $i \in [n]$. Goods are assigned **prices** $\boldsymbol{p} \in \mathbb{R}^m$. Note that prices are **anonymous**, in that all copies of good j are assigned the same price p_j.
An allocation is **feasible** iff

- no more than 1 unit of a good $j \in [m]$ is allocated in total, across all buyers, $\sum\limits_{i \in [n]} x_{ij} \leq 1$ and
- buyers do not spend more than their budget: i.e., $\boldsymbol{p} \cdot \boldsymbol{x}_i \leq b_i$, for all $i \in [n]$, where $\boldsymbol{x}_i$ denotes the row vector corresponding to the allocation of buyer i in the matrix $\boldsymbol{X}$.

An **outcome** is a pair $(\boldsymbol{X}, \boldsymbol{p})$ consisting of an allocation and prices respectively.

3.3 Equilibrium

An outcome $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ is **utility maximizing** iff no buyer would prefer a different feasible allocation of goods than theirs, at the outcome's prices, that is $\forall i \in [n], \boldsymbol{x}_i, \;\; u_i(\boldsymbol{x}_i^*) \geq u_i(\boldsymbol{x}_i)$.

For any outcome $(\boldsymbol{X}, \boldsymbol{p})$, we define the **demand** for good j as $\sum\limits_{i=1}^{n} x_{ij}$ and the **supply** of good j as 1 (since we assumed that there is 1 unit of each good).

An outcome $(\boldsymbol{X}, \boldsymbol{p})$ is **market clearing** iff the demand of each good is equal to its supply, i.e. $\sum\limits_{i=1}^{n} x_{ij} = 1$,

An outcome is called a **Walrasian equilibrium** iff it is feasible, utility maximizing and market clearing. If there are goods that buyers do not demand, then those goods must be priced at zero. i.e. $\forall i \in [n], \;\; \sum_{i \in [n]} x_{ij} = 1$ and $\sum_{j \in [m]} p_j (\sum_{i \in [n]} x_{ij} - 1)$

The existence of the equilibrium for any continuous utility functions can be shown through the use of Sperner's lemma[2]. Sperner's lemma is a combinatorial analog of Brouwer's fixed point theorem, which posits the existence of a fixed of a function in a very general setting. This proof however is non-constructive in that we cannot use it to compute an equilibrium outcome of the Fisher market. We will provide a convex program to compute an equilibrium for the Fisher Market for any continuous concave utility functions. This also is a constructive proof of the existence of an equilibrium for the Fisher Market for any continuous concave utility function since the convex program is guaranteed to have an optimal value.

4 Computing Equilbria: The Eisenberg-Gale Program

It turns out that the convex program that Edmund Eisenberg and Dave Gale provided for the Pari-mutuel Betting model provides a solution to Fisher Markets with linear utility functions[3]. While the primal Eisenberg-Gale program captures the equilibrium allocations, the dual of the program captures the equilibrium prices:

Primal

$$\max_{\boldsymbol{X}} \sum_{i=1}^{n} b_i \log\left(\sum_{j=1}^{m} v_{ij}x_{ij}\right) \tag{46}$$

$$\forall i \in [n], \quad \sum_{j=1}^{m} x_{ij} \leq 1 \tag{47}$$

$$\forall i \in [n], j \in [m] \quad x_{ij} \geq 0 \tag{48}$$

Dual

$$\min_{\boldsymbol{p},\boldsymbol{\beta}} \sum_{j=1}^{m} p_j - \sum_{i=1}^{n} b_i \log(\beta_i) \tag{49}$$

$$\forall i \in [n], j \in [m] \quad p_j \geq v_{ij}\beta_i \tag{50}$$

Notice that we can find any equilibrium allocation without knowing the prices (i.e., they are not part of the primal).

Let's give a meaning to these programs:

First note that Nash Social Welfare is defined as the budget weighted geometric mean of the utility of buyers that is:

$$SW_{NSW}(\boldsymbol{u};\boldsymbol{b}) = \left(\prod_{i\in[n]} u_i^{b_i}\right)^{\frac{1}{\sum_{i\in[n]} b_i}} \tag{51}$$

Primal: Maximizing the budget weighted log of utilities, is equivalent to maximizing the geometric mean of the utilities weighted by the budgets. This objective actually corresponds the Nash Social Welfare objective, which when maximized also maximizes the individual utilities constrained by the budget they are weighted by. The first

[2]https://en.wikipedia.org/wiki/Fisher_market#cite_note-3

constraint who associated dual variable are the prices is the market clearance condition, while the second constraint is simply the non-negativity of allocations (required for the boundedness of the program.

Dual: By the strong theorem of duality we know that the objective of the dual must be equal to the primal at the optimal values. This means that the objective function captures the optimal Nash Social Welfare. However as opposed to finding the allocation that maximizes Nash social welfare, in the dual we are instead trying to find values for prices (variables $\boldsymbol{p}$) and costs of unit of utility (variables $\boldsymbol{\beta}$) such that Nash Social welfare is minimized. One interesting thing to note is that the gradient of the objective function corresponds to excess demand which means that solving this program through gradient descent is equivalent to solving it with Tatonnement.

We now generalize the Eisenberg-Gale convex program so that it can provide a solution for any continuous concave utility functions and provide a proof that it provides a solution to the Fisher Market with continuous, concave, homogeneous utility functions .

Theorem 4.1. *Let n be the number of buyers and m be the number of goods in a Fisher market. If the utility function u_i for any consumer i is homogeneous of degree k $\left[\text{i.e., } \forall \boldsymbol{x} \in \mathbb{R}^m \;\; \lambda > 0, \quad u_i(\lambda \boldsymbol{x}) = \lambda^k u_i(\boldsymbol{x})\right]$ and concave $[\text{i.e., } \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m, \;\; \lambda \in (0,1), \quad u_i(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \lambda u_i(\boldsymbol{x}) + (1-\lambda)\boldsymbol{y}]$, then the following program computes equilibrium allocations:*

$$\max_{\boldsymbol{X}} \sum_{i=1}^{n} b_i \log\left(u_i\left(\boldsymbol{x}_i\right)\right) \tag{52}$$

$$\forall j \in [m], \qquad \sum_{i=1}^{n} x_{ij} \leq 1 \tag{53}$$

$$\forall i \in [n], j \in [m] \qquad x_{ij} \geq 0 \tag{54}$$

and the dual variable corresponding to constraint (53) *corresponds to the equilibrium prices associated with the equilibrium allocation determined by the program.*

Proof:

Given a homogenenous function of degree k, transform it to a homogeneous function of degree 1 by using a monotonic transformation which preserves the underlying preference relationship of the utility function. That is, any homogeneous function $f(x)$ of degree k, can be transformed to a homogeneous function of degree 1 using the monotonically non-decresing transformation $\sqrt[k]{f(x)}$. This allows us to conserve all properties of the utility function, more specifically the preference relations between goods by the increasing transformations theorem.

First, note that since the utilities are concave and the logarithm function is a concave function the objective function is also concave. Furthermore, as the constraints are all affine, the program we propose is feasible and bounded. We then write down the the Lagrangian L for this convex program, using slack variables $\forall j \in [m], \;\; p_j \geq 0$ and $\forall j \in [m], i \in [n], \;\; \lambda_{ij}$:

$$L(\boldsymbol{X}, \boldsymbol{p}, \boldsymbol{\lambda}) = \sum_{i=1}^{n} -b_i \log(u_i\left(\boldsymbol{x}_i\right)) + \sum_{j=1}^{m} p_j \left(\sum_{i=1}^{n} x_{ij} - 1\right) + \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_{ij}\left(-x_{ij}\right) \tag{55}$$

Any optimal solution of a convex program is guaranteed to satisfy a series of conditions called the Karush–Kuhn–Tucker

(KKT) conditions.[3] From the complementary slackness condition, we have the following two conditions:

$$p_j \left(\sum_{i \in [n]} x_{ij} - 1 \right) = 0 \tag{56}$$

$$\lambda_{ij} x_{ij} = 0 \tag{57}$$

From condition (56), we can also deduce the slack variable p_j corresponds to the equilibrium prices for the fisher market since this condition can be interpreted exactly as market clearance since it implies the following:

$$\forall j \in [m], \qquad \text{If } p_j > 0, \text{ then } \sum_{i \in [n]} x_{ij} = 1 \tag{58}$$

$$\forall j \in [m], \qquad \text{If } \sum_{i \in [n]} x_{ij} < 1 \text{ then } p_j = 0 \tag{59}$$

That is, the above conditions state that if the price of a good is positive then its supply must be equal to its demand and if a good is not entirely demanded then it is priced at zero.

Second, we will manipulate (60) to prove that consumers do not spend more than their budget. From first order KKT optimality conditions (i.e., the stationarity conditions), we get:

$$\nabla L = \frac{-b_i}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} + p_j - \lambda_{ij} = 0 \tag{60}$$

$$p_j = \frac{b_i}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} + \lambda_{ij} \tag{61}$$

Multiplying both sides by x_{ij} gives us:

$$p_j x_{ij} = \frac{b_i x_{ij}}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} + \lambda_{ij} x_{ij} \tag{62}$$

Using (57), we replace $\lambda_{ij} x_{ij}$ by 0:

$$p_j x_{ij} = \frac{b_i x_{ij}}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} \tag{63}$$

Summing up both sides across all goods, we obtain:

$$\sum_{j=1}^{m} p_j x_{ij} = b_i \sum_{j=1}^{m} \frac{x_{ij}}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} \tag{64}$$

Before completing the proof, we need to prove one more theorem.

[3] More background on this can be found here

Theorem 4.2. ***Euler's Theorem***
Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a homogeneous function of degree k that is continuous and differentiable on $\mathbb{R}^n_{>0}$, then the following holds:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(\boldsymbol{x}) x_i = k f(\boldsymbol{x}) \tag{65}$$

Proof: Assume that f is a homogeneous function of degree k. Let $\boldsymbol{x} \in \mathbb{R}^n_{>0}$. Define $g : (0, \infty) \to \mathbb{R}$ such that:

$$g(\lambda) = f(\lambda \boldsymbol{x}) - \lambda^k f(\boldsymbol{x}) \tag{66}$$

Due to f being homogeneous, this function has a value of 0 for its entire domain. This implies that its derivative is also 0 for its domain:

$$g'(\lambda) = 0 \tag{67}$$

Using the chain rule, we also know that the derivative of g can also be calculated as:

$$g'(\lambda) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(\boldsymbol{x}) x_i - k f(\boldsymbol{x}) \tag{68}$$

Using (67) and setting $\lambda = 0$, we then get:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(\boldsymbol{x}) x_i = k f(\boldsymbol{x}) \tag{69}$$

Going back to the proof of theorem 4.1, we have:

$$\sum_{j=1}^{m} p_j x_{ij} = b_i \sum_{j=1}^{m} \frac{x_{ij}}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} \tag{70}$$

$$\sum_{j=1}^{m} p_j x_{ij} = \frac{b_i}{u_i(\boldsymbol{x}_i)} \sum_{j=1}^{m} \frac{\partial}{\partial x_{ij}} u_i(\boldsymbol{x}_i) x_{ij} \tag{71}$$

(72)

Using Euler's theorem with $k = 1$ (since by our assumption the utility functions are homogeneous of degree 1) we get:

$$\sum_{j=1}^{m} p_j x_{ij} = \frac{b_i}{u_i(\boldsymbol{x}_i)} \sum_{j=1}^{m} \frac{\partial}{\partial x_{ij}} u_i(\boldsymbol{x}_i) x_{ij} \tag{73}$$

$$\sum_{j=1}^{m} p_j x_{ij} = \frac{b_i}{u_i(\boldsymbol{x}_i)} u_i(\boldsymbol{x}_i) \tag{74}$$

$$\sum_{j=1}^{m} p_j x_{ij} = b_i \tag{75}$$

Notice that the left hand side of this expression is exactly the spending of any buyer. This result implies that consumers are not spending more than their budget.

We will now show that the optimal allocation given by this program is utility maximizing. Recall from the stationarity condition (60) that we have:

$$\forall i \in [n], j \in [m], \qquad p_j = \frac{b_i}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} - \lambda_{ij} \tag{76}$$

$$\forall i \in [n], j \in [m], \qquad p_j \geq \frac{b_i}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} \tag{77}$$

$$\forall i \in [n], j \in [m], \qquad \frac{u_i(\boldsymbol{x}_i)}{b_i} \geq \frac{\frac{\partial u_i}{\partial x_{ij}}}{p_j} \tag{78}$$

The last inequality implies utility maximization as it means that the utility per buck of the bundle possessed by the buyer is greater than the utility per buck of any other item in the market. That is, if a buyer spent money on any good other than the current bundle he currently possesses he would get less utility since the utility per buck of every other good not in his current bundle will be less.

One thing to note is that while quasi-linear utilities are concave and continuous, we cannot use the above convex program to find an equilibrium outcome for the quasi-linear case of the Fisher Market. This is because quasi-linear utilities are not homogeneous and are parametrized by prices which then cause the primal of the above program to no more be convex. More recent work has shown that another convex program, namely Shmyrev's convex program can be used to calculate the equilibrium outcome for a fisher market with quasi-linear utilities. For the program we are going to propose we consider quasi-utility functions for which goods are perfect substitutes (as opposed to the combinatorial auction setting where the goods might have been interrelated and the value of a buyer for a bundle could not be determined by the value of the goods making up a bundle), that is:

$$\forall i \in [n], \qquad u_i = \sum_{j \in [m]} x_{ij} \left(v_{ij} - p_j \right) \tag{79}$$

Theorem 4.3. *The primal of the following program called Schmyrev's program captures the equilibrium allocations while the dual captures the equilibrium prices of a Fisher Market with quasilinear utilities[1]:*

Primal

$$\max_{\boldsymbol{X},\boldsymbol{u},\boldsymbol{v}} \sum_{i=1}^{n} b_i \log(u_i) - v_i \tag{80}$$

$$\forall i \in [n], \quad u_i \leq \sum_{j\in[m]} v_{ij}x_{ij} + v_i \tag{81}$$

$$\forall i \in [n], \quad \sum_{j\in[m]} x_{ij} \leq 1 \tag{82}$$

$$\forall i \in [n], j \in [m] \quad x_{ij}, v_i \geq 0 \tag{83}$$

Dual

$$\min_{\boldsymbol{p},\boldsymbol{\beta}} \sum_{j=1}^{m} p_j - \sum_{i=1}^{n} b_i \log(\beta_i) \tag{84}$$

$$\forall i \in [n], j \in [m] \quad p_j \geq v_{ij}\beta_i \tag{85}$$

$$\forall i \in [n], \quad \beta_i \leq 1 \tag{86}$$

Theorem 4.4. ***First Welfare Theorem - Fisher Market***
If $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ is a Walrasian equilibrium then it is also pareto-optimal.

Proof:
Let $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ be a Walrasian equilibrium. By way of contradiction, assume that there exists another outcome $(\boldsymbol{X}, \boldsymbol{p})$ for which we have $\forall i \in [n], \ \ u_i(\boldsymbol{x}_i) \geq u_i(\boldsymbol{x}_i^*)$ and $\exists i \in [n], \ \ u_i(\boldsymbol{x}_i) > u_i(\boldsymbol{x}_i^*)$. Since utility functions are non-satiated and Walrasian equilibria are utility maximizing, then we must have that $\forall i \in [n], \ \ \boldsymbol{p} \cdot \boldsymbol{x}_i \geq \boldsymbol{p} \cdot \boldsymbol{x}_i^*$ and $\exists i \in [n], \ \ \boldsymbol{p} \cdot \boldsymbol{x}_i > \boldsymbol{p} \cdot \boldsymbol{x}_i^*$. That is in order to achieve a higher utility with another allocation that is not a Walrasian equilibrium, buyers need to be on an indifference curve that intersects with a budget constraint curve further way from the origin. Since at a Walrasian equilibrium buyers already were spending their entire budget, any outcome that pareto-dominates the Walrasian Equilibrium outcome $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ must be infeasible.

Theorem 4.5. ***Second Welfare Theorem - Fisher Market***
Let $(\boldsymbol{u}, \boldsymbol{b})$ be a Fisher Market, if $\boldsymbol{X}^*$ is pareto-optimal allocation for $(\boldsymbol{u}, \boldsymbol{b})$ then there exists a price vector $\boldsymbol{p}^*$ such that $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ is a Walrasian equilibrium. That is, any Pareto-optimal outcome can be represented as Walrasian outcome.

Skipping proof as it is relatively involved.

Note that the second welfare theorem does not say that for any Fisher market, every Pareto optimal allocation is a Walrasian equilibrium. Rather, it says that for any Pareto optimal allocation of a Fisher Market there is a way to re-distribute resources through prices that makes the allocation a Walrasian equilibrium outcome.

5 Summary

In this section, we summarize some commonalities and differences of programs for computing Walrasian Equilibria in the indivisible and divisible setting.

First, we note that in all the programs we have studied, the primal aims to maximize a certain welfare function. While in the combinatorial auction setting (indivisble + no budgets) this welfare objective is utilitarian welfare, in the Fisher Market setting (divisible + budgets), the welfare objective is Nash Social Welfare.

Second, prices always correspond to the dual variable corresponding to the market clearance condition in the primal.

Third, while in the divisible setting we are guaranteed the existence of Walrasian equilbria and we can always compute them through convex programming, in the indivisible setting, we are guaranteed an integer solution through linear programming iff a Walrasian equilbrium exists (which we are not guaranteed that it exists).

Finally, in the indivisible setting Walrasian equilbria are more powerful, namely they are not only pareto-optimal but also welfare maximizing. In the divisible setting Walrasian equilbria are only pareto-optimal. As a result, in the indivisible setting we maximize utilitarian welfare in the primal to get equilibrium allocations since Walrasian equilbiria are welfare-maximizing and in the inidivisble setting maximizing welfare guarantees utility maximization. In the divisible setting, we instead maximize nash social welfare which allows utility maximization and combined with the market clearance condition gives us pareto-optimal allocations.

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