Wagering and Fair Division Models: Parimutuel and Fisher Markets

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1 Parimutuel Model

In 1867, Spanish entrepreneur Joseph Oller invented parimutuel betting, a form of wagering that is still popular today and handles billions of waged dollars every year! The parimutuel model is a system of betting in which all bets are placed on a set of possible exclusive outcomes. Once the outcome is determined, the bettors who bet on the winning outcome split the total amount wagered in proportion to the size of their wagers. To explain this system better we will consider the application of parimutuel betting to betting in horse races.

In this setting, each bettor bets on a horse. The house collects the bets and pays the participants that bet on the winning horse the total amount of money collected multiplied by the proportion of each winner's bet in the total amount bet on the winning horse.

The first principle of the parimutuel model is that bettors should not try to bet on the horse that has the highest chance of winning, but on the horse that has the best pay out ratio relative to your belief of its chance of winning. The reason for this is simply that as more people bet on the same horse the likelihood of any bettor making a profit goes to 0. To better illustrate this principle, consider the trivial case when every agent bets all their money on the same horse. If that horse ends up winning, every bettor gets back the amount of money he/she bet but makes no profit whatsoever. Since, the goal of betting is to make a profit, a bettor is better off guessing a winning horse on which only a small amount of money is bet. With this in mind, we would now like to model the parimutuel betting model as a game so that we can analyze and better understand the parimutuel betting system. In particular, we want to answer the following question, given the true beliefs of bettors for each horse, can we predict the bets of the bettors as they would happen in the real world?

1.1 Model

We now introduce the mathematical model proposed by Edmund Eisenberg and David Gale:[1] to model parimutuel betting. The model consists of n bettors and m horses. Every bettor $i \in [n]$, has a budget $b_i \in \mathbb{R}$ and a vector of *subjective opinions* $\boldsymbol{p}_i \in \mathbb{R}^m$ whose j^{th} entry denotes the belief of the bettor i that horse j will win. Without loss of generality, we assume that the sum of the budgets is equal to one, i.e., $\sum_{i \in [n]} b_i = 1$. We denote the subjective opinions of all bettors by the matrix $\boldsymbol{P} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)^T \in \mathbb{R}^{n \times m}$. Without loss of generality, we assume that each horse j has at least one subjective opinion of winning that is strictly positive, i.e. $p_{ij} > 0$. Otherwise, we could simply not consider that horse in the betting process because no one will bet money on it (since no bettor believes that the horse will win).

[1] Fun fact: Gale was a Professor in the applied math department here at Brown!

1.2 Outcome of the Model

Each bettor $i \in [n]$ bets part of his budget on some horse $j \in [m]$. Let $\boldsymbol{\beta} \in \mathbb{R}^{n \times m}$ be the **bet allocation** matrix whose $(i,j)^{th}$ entry describes the amount that bettor i bet on horse j. After the bets have been made, all bettors' opinions are aggregated in order to decide the perceived overall winning probability of each horse. Let $\boldsymbol{\pi} \in \mathbb{R}^m$ be the **final track probabilities** vector, which represents the probability of each horse winning based on the bets that have been made.[2] A parimutuel outcome is a tuple $(\boldsymbol{\beta}, \boldsymbol{\pi})$, consisting of bet allocations and final track probabilities.

Any parimutuel outcome must satisfy the **budget relation**: $\sum_{j \in [m]} \beta_{ij} = b_i$ (i.e., the sum of the bets of each bettor is equal to their budget) and the **parimutuel condition**: $\sum_{i \in [n]} \beta_{ij} = \pi_j$ (i.e., the sum of the bets on each horse is equal to their final track probability). The parimutuel condition simply describes the main principle behind the model, which is that the probability of a horse winning is proportional to the amount of money bet on that horse. Note that since we assumed that the sum of all of the budgets is equal to 1, in this case the proportonality constant is exactly 1, i.e., the π_j's represent exact probabilities.

As stated before, the goal of each bettor is to bet on the horse that has the best payout ratio relative to their belief of the horse's winning odds. Since the bettors make a bigger profit when the winning horse has less money bet on it, the goal of each bettor can be modelled as betting on the horses with the highest ratio of subjective probability of winning to final track probability[3]. A parimutuel outcome $(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*)$ in which the bettors bet in this manner is said to be **expectation maximizing**. We arithmetize the expectation maximization condition as follows:

$$\text{if } \mu_i = \max_s \frac{p_{is}}{\pi_s^*} \text{ and } \beta_{ij}^* > 0, \text{ then } \mu_i = \frac{p_{ij}}{\pi_j^*} \tag{1}$$

The above expression simply says that if a bettor bets on a certain horse, then that horse must have the highest ratio of subjective winning probability to final track probability.

A parimutuel outcome is **optimal** iff bettors are expectation maximizing and the outcome respects the budget relation as well as the parimutuel condition. We should note that the final track probabilities cannot be determined before the bettors have put their money on their horses. However, bettors can also not maximize their expectation without knowing the final track probabilities! So one question to ask is whether if an optimal outcome of the parimutuel model exists at all!

1.3 Optimal Solution

We now introduce the **Eisenberg-Gale convex program**:

$$\max_{\boldsymbol{\xi}} \sum_{i \in [n]} b_i \log \left(\sum_{j \in [m]} p_{ij} \xi_{ij} \right) \tag{2}$$

$$\sum_{i \in [n]} \xi_{ij} = 1 \tag{3}$$

$$\xi_{ij} \geq 0 \tag{4}$$

$$\tag{5}$$

The optimal output $\bar{\boldsymbol{\xi}} \in \mathbb{R}^{n \times m}$ to this convex program can be used to calculate the optimal allocation $(\boldsymbol{\beta}, \boldsymbol{\pi})$. This is also a proof of the existence an optimal allocation for the parimutuel problem (given that our claim in the previous sentence is correct), since any feasible convex program is guaranteed to have a minimum.

[2]This vector of probability represents the aggregation of all bettors subjective probabilities into a unique probability vector.

[3]Remember that a lower final track probability means less money bet on a horse due to the parimutuel condition

Theorem 1.1. *Let $\bar{\boldsymbol{\xi}}$ be a solution of the Eisenberg-Gale program. An optimal allocation $(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*)$ for the parimutuel market can be calculated as:*

$$\pi_j^* = \max_i \frac{b_i p_{ij}}{\sum\limits_{s\in[m]} p_{is}\bar{\xi}_{is}} \tag{6}$$

$$\beta_{ij}^* = \bar{\xi}_{ij}\pi_j^* \tag{7}$$

Proof: First, we claim the following, which we use in the rest of the proof:

$$\text{if } \bar{\xi}_{ij} > 0 \text{ then } \pi_j = \frac{b_i p_{ij}}{\sum\limits_{s\in[m]} p_{ij}\bar{\xi}_{is}} \tag{8}$$

To see this, suppose that this claim is false and that for some i, j, $\bar{\xi}_{ij} > 0$ and that $\pi_j > \frac{b_i p_{ij}}{\sum\limits_{s\in[m]} p_{ij}\bar{\xi}_{is}}$. By definition of π_j, we then have a $\bar{\xi}_{kj}$ for some k which gives $\pi_j = \frac{b_k p_{kj}}{\sum\limits_{s\in[m]} p_{ks}\bar{\xi}_{ks}} > \frac{b_i p_{ij}}{\sum\limits_{s\in j} p_{ij}\bar{\xi}_{is}}$. This then implies that we could decrease $\bar{\xi}_{ij}$ and increase $\bar{\xi}_{kj}$ to increase the objective function of the Eisenberg-Gale program. This, however, is a contradiction since $\bar{\xi}$ maximizes the Eisenberg-Gale program. In other words, this simply comes from the fact that $\bar{\xi}$ maximizes the Eisenberg-Gale program.

With this fact in mind, we now prove that the solution we have given satisfies the optimality conditions, namely 1) the budget constraint, 2) the parimutuel condition, and 3) expectation maximization.

Budget constraint condition: Combining conditions (7) and (8), we get:

$$\beta_{ij}^* = \bar{\xi}_{ij}\pi_j^* \tag{9}$$

$$= \bar{\xi}_{ij}\frac{b_i p_{ij}}{\sum\limits_{s\in[m]} p_{ij}\bar{\xi}_{is}} \tag{10}$$

$$= b_i \frac{p_{ij}\bar{\xi}_{ij}}{\sum\limits_{s\in[m]} p_{ij}\bar{\xi}_{is}} \tag{11}$$

Summing the above on j, we get: $\sum_{j\in[m]} \beta_{ij}^* = b_i \frac{\sum\limits_{j\in[m]} p_{ij}p_{ij}\bar{\xi}_{ij}}{\sum\limits_{s\in[m]} p_{ij}\bar{\xi}_{is}} = b_i$. This confirms the budget constraint condition.

Parimutuel Condition: By constraint (3), we know that $\sum_{i\in i} \bar{\xi}_{ij} = 1$. Hence, summing up β_{ij}^* on i we recover the final track probabilities, i.e., $\sum_{i\in[n]} \beta_{ij}^* = \sum_{i\in[n]} \bar{\xi}_{ij}\pi_j^* = \pi_j^* \sum_{i\in[n]} \bar{\xi}_{ij} = \pi_j^*$. This confirms that the allocation respects the parimutuel condition.

Expectation Maximization

From (6), we know that:

$$\pi_j^* = \max_i \frac{b_i p_{ij}}{\sum\limits_{s\in[m]} p_{is}\bar{\xi}_{is}} \tag{12}$$

$$\frac{1}{\pi_j^*} = \min_i \frac{\sum\limits_{s\in[m]} p_{is}\bar{\xi}_{is}}{b_i p_{ij}} \tag{13}$$

$$\frac{p_{ij}}{\pi_j^*} = \min_i \frac{\sum\limits_{s\in[m]} p_{ij}\bar{\xi}_{is}}{b_i} \tag{14}$$

$$\frac{p_{ij}}{\pi_j^*} \leq \frac{\sum\limits_{s\in[m]} p_{is}\bar{\xi}_{is}}{b_i} \tag{15}$$

Recall from (1) that:

$$\mu_i = \max_s \frac{p_{is}}{\pi_s^*} \tag{16}$$

Substituting π_s^* with the expression from (6), we get:

$$\mu_i = \max_s p_{ij} \frac{\sum\limits_{k\in[m]} p_{ij}\bar{\xi}_{ik}}{b_i p_{ij}} \tag{17}$$

$$= \max_s \frac{\sum\limits_{k\in[m]} p_{ij}\bar{\xi}_{ik}}{b_i} \tag{18}$$

$$= \frac{\sum\limits_{k=1}^{n} p_{ik}\bar{\xi}_{ik}}{b_i} \tag{19}$$

Since we assumed that there exists at least one entry in each column of $\boldsymbol{P}$ that is strictly positive, we know that each π_j is positive[4] Then, combining facts (7), (18), (15) and (8), we get exactly (1).

[4]Observe the objective function of the Eisenberg-Gale program to convince yourself about this fact.

2 Fisher Market Model

The Fisher market model is the first model of a market that proves that a competitive equilibrium exists in a market, that is, there exists prices and allocation of goods that maximize the utility of buyers and clear the market. In other words, the Fisher market proves that in a market, prices exist such that buyers have gotten their the supply is equal to demand.

2.1 Market Elements

The Fisher market model consists of a finite set of n heterogeneous **buyers** $[n]$ and a finite set of m heterogeneous divisible **good** types $[m]$. Each buyer $i \in [n]$ is characterized by a budget $b_i \in \mathbb{R}$ and a utility function, $u_i : \mathbb{R}^m \to \mathbb{R}_+$, giving the utility that buyer i derives from any bundle of goods. Without loss of generality, we assume that budgets are normalized, i.e., $\sum_{i=1}^n b_i = 1$, and that there is only one unit of each good $j \in [m]$. The results/solutions we provide in this section can be applied to the more general settings in which the number of copies of each good is different than 1 and there are goods that are not demanded.

2.2 Outcome of the Market

An **allocation** $\boldsymbol{X}$ is a map from goods to buyers, represented as a matrix, s.t. $x_{ij} \geq 0$ denotes the amount of good $j \in [m]$ allocated to buyer $i \in [n]$. Goods are assigned **prices** $\boldsymbol{p} \in \mathbb{R}^m$. Note that prices are **anonymous**, in that copies of the same good are assigned the same price. p_j.
An allocation is **feasible** iff buyers do not spend more than their budget: i.e., $\boldsymbol{p} \cdot \boldsymbol{x}_i \leq b_i$, for all $i \in [n]$, where $\boldsymbol{x}_i$ denotes the row vector corresponding to the allocation of buyer i in the matrix $\boldsymbol{X}$.

An **outcome** of the Fisher market is a tuple $(\boldsymbol{X}, \boldsymbol{p})$ consisting of an allocation and prices, respectively.

2.3 Equilibrium

An outcome $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ is **utility maximizing** iff no buyer would strictly prefer a different feasible allocation of goods than the one they are allocated at the outcome's prices, or $\forall i \in [n], \quad \boldsymbol{x}_i^* \in \arg\max_{\boldsymbol{x}\cdot\boldsymbol{p}^* \leq b_i} u_i(\boldsymbol{x}_i)$. That is, a utility maximizing outcome consists of an allocation that maximizes the utility function of the buyers at the given prices constrained by their budget.

For any outcome $(\boldsymbol{X}, \boldsymbol{p})$, we define the **demand** for good j as $\sum\limits_{i=1}^{n} x_{ij}$ and the **supply** of good j as 1 (since we assumed that there is 1 unit of each good).

An outcome $(\boldsymbol{X}, \boldsymbol{p})$ is **market clearing** iff the demand of each good is less than or equal to its supply, i.e. $\sum\limits_{i=1}^{n} x_{ij} \leq 1$, and goods whose supply are greater than their demand are priced at zero, i.e., $\forall j \in [m] \sum_{j\in[m]} p_j(\sum\limits_{i=1}^{n} x_{ij} - 1) = 0$. Note that the second part of the market clearance condition is called **Walras' law**.

An outcome is called a **Walrasian equilibrium** iff it is 1) feasible, 2) utility maximizing and 3) market clearing.

The existence of a Walrasian equilibrium for any continuous, quasi-concave and non-satiated utility functions can be established through the use of Sperner's lemma[5]. Sperner's lemma is a combinatorial analog of Brouwer's fixed point theorem, which posits the existence of a fixed of a function in a very general setting. This proof however is non-constructive in that we cannot use it to compute an equilibrium outcome of the Fisher market. We will provide

[5]https://en.wikipedia.org/wiki/Fisher_market#cite_note-3

later a convex program to compute an equilibrium for the Fisher Market for any continuous concave utility functions. This also is a constructive proof of the existence of an equilibrium for the Fisher Market for any continuous concave utility function since the convex program is guaranteed to have an optimal value. Before we propose such a convex program however, we first establish an equivalence between the Parimutuel market model and the Fisher market model, which will allow us to derive a convex program for the Fisher market.

3 Connecting the Parimutuel Model and the Fisher Market

We will now assume that the utility functions of the buyers are linear to provide an interesting connection between the optimal outcome of the Parimutuel model and the Fisher Market. Let each buyer have preferences over goods represented as a linear utility function that is parametrized by the vector of values for each buyer $i \in [n]$ $\boldsymbol{v}_i \in \mathbb{R}^m$:

$$\forall i \in [n], u_i(\boldsymbol{x}_i) = \sum_{j \in [m]} v_{ij} x_{ij} \tag{20}$$

It turns out that any equilibrium allocation of the fisher market is captured by the solution to the Eisenberg-Gale program. The correspondence between the variables in the both models is given in the table below:

Parimutuel	**Fisher**
b_i (bettor budget)	b_i (buyer budget)
ξ_{ij} (proportion of i's bet in π_j)	X_{ij} (allocation of good j to buyer i)
p_{ij} (subjective probability)	v_{ij} (valuation)
π_j (final track probability)	p_j (price)
β_{ij} (bet of B_i on H_j)	$X_{ij}p_j$ (spending of buyer i on good j)

The one-to-one correspondence between the Parimutuel market model and the Fisher market model implies that that Eisenberg-Gale program captures the equilibrium allocations of the linear Fisher market, and the dual of the program captures the equilibrium prices of the Fisher market. In other words, equilibria in the Parimutuel market correspond with equilibria in the Fisher market, and we can find both of them using the Eisenberg-Gale program.

4 Computing Equilbria: The Eisenberg-Gale Program

As mentioned before, it turns out that the convex program that Edmund Eisenberg and Dave Gale provided for the Parimutuel Betting model provides a solution to Fisher Markets with linear utility functions[1]. While the primal Eisenberg-Gale program captures the equilibrium allocations, the dual of the program captures the equilibrium prices:

Primal

$$\max_{\boldsymbol{X}} \sum_{i=1}^{n} b_i \log\left(\sum_{j=1}^{m} v_{ij} x_{ij}\right) \tag{21}$$

$$\forall i \in [n], \qquad \sum_{j=1}^{m} x_{ij} \leq 1 \tag{22}$$

$$\forall i \in [n], j \in [m] \qquad x_{ij} \geq 0 \tag{23}$$

Dual

$$\min_{\boldsymbol{p},\boldsymbol{\beta}} \sum_{j=1}^{m} p_j - \sum_{i=1}^{n} b_i \log(\beta_i) \tag{24}$$

$$\forall i \in [n], j \in [m] \qquad p_j \geq v_{ij}\beta_i \tag{25}$$

Notice that we can find any equilibrium allocation without knowing the prices (i.e., they are not part of the primal).

Let's give a meaning to these programs:

First note that Nash Social Welfare is defined as the budget weighted geometric mean of the utility of buyers:

$$SW_{NSW}(\boldsymbol{u}; \boldsymbol{b}) = \left(\prod_{i\in[n]} u_i^{b_i}\right)^{\frac{1}{\sum_{i\in[n]} b_i}} \tag{26}$$

Primal: Maximizing the budget weighted log of utilities, is equivalent to maximizing the geometric mean of the utilities weighted by the budgets. This objective actually corresponds the Nash Social Welfare objective, which when maximized, it also maximizes the individual utilities constrained by the budget they are weighted by. The first constraint ensures feasibility and as we will see will also imply market clearance, while the second constraint is simply the non-negativity of allocations (required for the boundedness of the program.

Dual: By the strong theorem of duality, we know that the objective of the dual must be equal to the primal at the optimal values. This means that the objective function captures the optimal Nash Social Welfare. However as opposed to finding the allocation that maximizes Nash social welfare, in the dual we are instead trying to find values for prices (variables $\boldsymbol{p}$) and costs of units of utility (variables $\boldsymbol{\beta}$) such that Nash Social welfare is minimized. One interesting thing to note is that the gradient of the objective function corresponds to excess demand, which means that solving this program through gradient descent is equivalent to solving it with Tatonnement.

We now generalize the Eisenberg-Gale convex program so that it can provide a solution for any continuous, concave utility functions, and we provide a proof that it gives a solution to the Fisher Market with continuous, concave, homogeneous utility functions.

Theorem 4.1. *Let n be the number of buyers and m be the number of goods in a Fisher market. If the utility function u_i for any consumer i is homogeneous of degree k $\left[\text{i.e., } \forall \boldsymbol{x} \in \mathbb{R}^m \ \ \lambda > 0, \quad u_i(\lambda \boldsymbol{x}) = \lambda^k u_i(\boldsymbol{x})\right]$ and concave $[\text{i.e., } \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m, \ \ \lambda \in (0,1), \quad u_i(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \lambda u_i(\boldsymbol{x}) + (1-\lambda)\boldsymbol{y}]$, then the following program computes equilibrium allocations:*

$$\max_{\boldsymbol{X}} \sum_{i=1}^{n} b_i \log\left(u_i\left(\boldsymbol{x}_i\right)\right) \tag{27}$$

$$\forall j \in [m], \qquad \sum_{i=1}^{n} x_{ij} \leq 1 \tag{28}$$

$$\forall i \in [n], j \in [m] \qquad x_{ij} \geq 0 \tag{29}$$

and the dual variable corresponding to constraint (28) *corresponds to the equilibrium prices associated with the equilibrium allocation determined by the program.*

Proof:

Given a homogenenous function of degree k, transform it to a homogeneous function of degree 1 by using a monotonic transformation which preserves the underlying preference relationship of the utility function. That is, any homogeneous function $f(x)$ of degree k, can be transformed to a homogeneous function of degree 1 using the monotonically non-decresing transformation $\sqrt[k]{f(x)}$. This allows us to conserve all properties of the utility function, more specifically the preference relations between goods by the increasing transformations theorem.

First, note that since the utilities are concave and the logarithm function is a concave function, the objective function must also be concave. Furthermore, as the constraints are all affine, the program we propose is feasible and bounded. We then write down the the Lagrangian L for this convex program, using slack variables, which is $\forall j \in [m], \ p_j \geq 0$ and $\forall j \in [m], i \in [n], \ \ \lambda_{ij}$:

$$L(\boldsymbol{X}, \boldsymbol{p}, \boldsymbol{\lambda}) = \sum_{i=1}^{n} -b_i \log(u_i\left(\boldsymbol{x}_i\right)) + \sum_{j=1}^{m} p_j \left(\sum_{i=1}^{n} x_{ij} - 1\right) + \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_{ij}\left(-x_{ij}\right) \tag{30}$$

Any optimal solution of a convex program is guaranteed to satisfy a series of conditions called the Karush–Kuhn–Tucker (KKT) conditions.[6] From the complementary slackness condition, we get the following two conditions:

$$p_j \left(\sum_{i \in [n]} x_{ij} - 1\right) = 0 \tag{31}$$

$$\lambda_{ij} x_{ij} = 0 \tag{32}$$

From condition (31), we can deduce that the slack variable p_j corresponds to the equilibrium prices for the Fisher market, since this condition can be interpreted exactly as market clearance since it implies the following:

[6] More background on this can be found here

$$\forall j \in [m], \qquad \text{If } p_j > 0, \text{ then } \sum_{i \in [n]} x_{ij} = 1 \tag{33}$$

$$\forall j \in [m], \qquad \text{If } \sum_{i \in [n]} x_{ij} < 1 \text{ then } p_j = 0 \tag{34}$$

That is, the above conditions state that if the price of a good is positive, then its supply must be equal to its demand, and if a good is not entirely demanded, then it is priced at zero.

Second, we will manipulate (35) to prove that consumers do not spend more than their budget. From first order KKT optimality conditions (i.e., the stationarity conditions), we get:

$$\nabla L = \frac{-b_i}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} + p_j - \lambda_{ij} = 0 \tag{35}$$

$$p_j = \frac{b_i}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} + \lambda_{ij} \tag{36}$$

Multiplying both sides by x_{ij} gives us:

$$p_j x_{ij} = \frac{b_i x_{ij}}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} + \lambda_{ij} x_{ij} \tag{37}$$

Using (32), we replace $\lambda_{ij} x_{ij}$ by 0:

$$p_j x_{ij} = \frac{b_i x_{ij}}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} \tag{38}$$

Summing up both sides across all goods, we obtain:

$$\sum_{j=1}^{m} p_j x_{ij} = b_i \sum_{j=1}^{m} \frac{x_{ij}}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} \tag{39}$$

Before completing the proof, we need to prove one more theorem.

Theorem 4.2. ***Euler's Theorem***

Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a homogeneous function of degree k that is continuous and differentiable on $\mathbb{R}^n_{>0}$, then the following holds:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(\boldsymbol{x}) x_i = k f(\boldsymbol{x}) \tag{40}$$

Proof: Assume that f is a homogeneous function of degree k. Let $\boldsymbol{x} \in \mathbb{R}^n_{>0}$. Define $g : (0, \infty) \to \mathbb{R}$ such that:

$$g(\lambda) = f(\lambda \boldsymbol{x}) - \lambda^k f(\boldsymbol{x}) \tag{41}$$

Due to f being homogeneous, this function has a value of 0 for its entire domain. This implies that its derivative is also 0 for its domain:

$$g'(\lambda) = 0 \tag{42}$$

Using the chain rule, we also know that the derivative of g can also be calculated as:

$$g'(\lambda) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(\boldsymbol{x}) x_i - k f(\boldsymbol{x}) \tag{43}$$

Using (42) and setting $\lambda = 0$, we then get:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(\boldsymbol{x}) x_i = k f(\boldsymbol{x}) \tag{44}$$

Going back to the proof of theorem 4.1, we have:

$$\sum_{j=1}^{m} p_j x_{ij} = b_i \sum_{j=1}^{m} \frac{x_{ij}}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} \tag{45}$$

$$\sum_{j=1}^{m} p_j x_{ij} = \frac{b_i}{u_i(\boldsymbol{x}_i)} \sum_{j=1}^{m} \frac{\partial}{\partial x_{ij}} u_i(\boldsymbol{x}_i) x_{ij} \tag{46}$$

(47)

Using Euler's theorem with $k = 1$ (since by our assumption the utility functions are homogeneous of degree 1) we get:

$$\sum_{j=1}^{m} p_j x_{ij} = \frac{b_i}{u_i(\boldsymbol{x}_i)} \sum_{j=1}^{m} \frac{\partial}{\partial x_{ij}} u_i(\boldsymbol{x}_i) x_{ij} \tag{48}$$

$$\sum_{j=1}^{m} p_j x_{ij} = \frac{b_i}{u_i(\boldsymbol{x}_i)} u_i(\boldsymbol{x}_i) \tag{49}$$

$$\sum_{j=1}^{m} p_j x_{ij} = b_i \tag{50}$$

Notice that the left hand side of the above expression is exactly the spending of any buyer. This result implies that consumers are not spending more than their budget.

We will now show that the optimal allocation given by this program is utility maximizing. Recall from the stationarity condition (35) that we have:

$$\forall i \in [n], j \in [m], \qquad p_j = \frac{b_i}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} - \lambda_{ij} \tag{51}$$

$$\forall i \in [n], j \in [m], \qquad p_j \geq \frac{b_i}{u_i(\boldsymbol{x}_i)} \frac{\partial u_i}{\partial x_{ij}} \tag{52}$$

$$\forall i \in [n], j \in [m], \qquad \frac{u_i(\boldsymbol{x}_i)}{b_i} \geq \frac{\frac{\partial u_i}{\partial x_{ij}}}{p_j} \tag{53}$$

The last inequality implies utility maximization, since it means that the utility per buck of the bundle possessed by the buyer is greater than the utility per buck of any other item in the market. That is, if a buyer spent money on any good other than the current bundle he currently possesses, he would get less utility since the utility per buck of every other good not in his current bundle is smaller than those in his bundle.

One thing to note is that while quasi-linear utilities are concave and continuous, we cannot use the above convex program to find an equilibrium outcome for the quasi-linear case of the Fisher Market. This is because quasi-linear utilities are not homogeneous and are parametrized by prices that cause the primal of the above program to not be convex. More recent work has shown that another convex program, namely Shmyrev's convex program, can be used to calculate the equilibrium outcome for a Fisher market with quasi-linear utilities.

For the program we are going to propose we consider quasilinear utility functions for which goods are perfect substitutes (as opposed to the combinatorial auction setting where the goods might have been interrelated and the value of a buyer for a bundle could not be determined by the value of the goods making up a bundle), that is:

$$\forall i \in [n], \qquad u_i = \sum_{j \in [m]} x_{ij} \left(v_{ij} - p_j \right) \tag{54}$$

Theorem 4.3. *The primal of the following program called Schmyrev's program captures the equilibrium allocations, while the dual captures the equilibrium prices of a Fisher Market with quasilinear utilities [2]:*

Primal

$$\max_{\boldsymbol{X},\boldsymbol{u},\boldsymbol{v}} \sum_{i=1}^{n} b_i \log(u_i) - v_i \tag{55}$$

$$\forall i \in [n], \quad u_i \leq \sum_{j\in[m]} v_{ij}x_{ij} + v_i \tag{56}$$

$$\forall i \in [n], \quad \sum_{j\in[m]} x_{ij} \leq 1 \tag{57}$$

$$\forall i \in [n], j \in [m] \quad x_{ij}, v_i \geq 0 \tag{58}$$

Dual

$$\min_{\boldsymbol{p},\boldsymbol{\beta}} \sum_{j=1}^{m} p_j - \sum_{i=1}^{n} b_i \log(\beta_i) \tag{59}$$

$$\forall i \in [n], j \in [m] \quad p_j \geq v_{ij}\beta_i \tag{60}$$

$$\forall i \in [n], \quad \beta_i \leq 1 \tag{61}$$

Theorem 4.4. ***First Welfare Theorem - Fisher Market***
If $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ is a Walrasian equilibrium, then it is also Pareto-optimal.

Proof:
Let $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ be a Walrasian equilibrium. By way of contradiction, assume that there exists another outcome $(\boldsymbol{X}, \boldsymbol{p})$ for which we have $\forall i \in [n], \;\; u_i(\boldsymbol{x}_i) \geq u_i(\boldsymbol{x}_i^*)$ and $\exists i \in [n], \;\; u_i(\boldsymbol{x}_i) > u_i(\boldsymbol{x}_i^*)$. Since utility functions are non-satiated and Walrasian equilibria are utility maximizing, then we must have that $\forall i \in [n], \;\; \boldsymbol{p} \cdot \boldsymbol{x}_i \geq \boldsymbol{p} \cdot \boldsymbol{x}_i^*$ and $\exists i \in [n], \;\; \boldsymbol{p} \cdot \boldsymbol{x}_i > \boldsymbol{p} \cdot \boldsymbol{x}_i^*$. That is in order to achieve a higher utility with another allocation that is not a Walrasian equilibrium, buyers need to be on an indifference curve that intersects with a budget constraint curve further way from the origin. Since at a Walrasian equilibrium buyers already were spending their entire budget, any outcome that pareto-dominates the Walrasian Equilibrium outcome $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ must be infeasible.

Theorem 4.5. ***Second Welfare Theorem - Fisher Market***
Let $(\boldsymbol{u}, \boldsymbol{b})$ be a Fisher Market, if $\boldsymbol{X}^*$ is pareto-optimal allocation for $(\boldsymbol{u}, \boldsymbol{b})$ then there exists a price vector $\boldsymbol{p}^*$ such that $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ is a Walrasian equilibrium. That is, any Pareto-optimal outcome can be represented as Walrasian outcome.

The second welfare theorem does not say that for any Fisher market, every Pareto optimal allocation is a Walrasian equilibrium. Rather, it says that for any Pareto optimal allocation of a Fisher Market there is a way to re-distribute resources through prices that makes the allocation a Walrasian equilibrium outcome.

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