Introduction to General Equilibrium Theory for Computer Scientists

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Contents

1 Preliminaries

We use Roman uppercase letters to denote sets, e.g., S. We us bold uppercase letters to denote matrices or tensors, e.g., $\boldsymbol{X}$, and bold lowercase letters to denote vectors, e.g., $\boldsymbol{p}$, and Roman lowercase letters to denote scalar quantities, e.g., c. We use a subscript i to denote the i^{th} component of a tensor, e.g. $\boldsymbol{X}_i$, or the i^{th} row vector of a matrix by the equivalent bold lowercase letter with subscript i (e.g., $\boldsymbol{x}_i$) Similarly, we denote the j^{th} entry of a vector (e.g., $\boldsymbol{p}$ or $\boldsymbol{x}_i$) by the equivalent Roman lowercase letter with subscript j (e.g., p_j or x_{ij}). We denote the set of numbers $\{1, \dots, n\}$ by $[n]$, the set of natural numbers by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$, the set of non-negative real numbers by $\mathbb{R}_+$ and the set of strictly positive real numbers by $\mathbb{R}_{++}$.

We also define some set operations. Unless otherwise stated, the sum of a scalar by a set and of two sets is defined as the Minkowski sum, e.g., $c + A = \{c + a \mid a \in A\}$ and $A + B = \{a + b \mid a \in A, b \in B\}$, and the product of a scalar by a set and two sets is defined as the Minkowski product, e.g., $cA = \{ca \mid a \in A\}$ and $AB = \{ab \mid a \in A, b \in B\}$.

2 Utilities

2.1 Preference Relations & Utility Functions

An important feature of economic markets is that they aggregate the preferences of consumers to what we know as prices. As a result, in order to define clear market model, we have to understand clearly the theory of utility functions.

Suppose an agent chooses from a set of goods $G = \{1, 2, 3, \dots\}$. For example, one can think of these goods as different TV sets or cars. Given two goods, $x \in G$ and $y \in G$:

- the agent weakly prefers x over y if x is at least as good as y. To avoid us having to write "weakly prefers" repeatedly, we simply write $x \succcurlyeq y$.

- the agent strongly prefers x over y if x is better than y. To avoid us having to write "strongly prefers" repeatedly, we simply write $x \succ y$.

We now put some basic structure on the agent's preferences by adopting two axioms.

Axiom 2.1 (Completeness Axiom). *A preference relation $\succcurlyeq$ on G is **complete** if for every pair $x, y \in X$, either $x \succcurlyeq y, y \succcurlyeq x$, or both.*

Axiom 2.2 (Transitivity Axiom). *A preference relation $\succcurlyeq$ on G is **transitive** if for every triple $x, y, z \in X$, if $x \succcurlyeq y$ and $y \succcurlyeq z$ then $x \succcurlyeq z$*

An agent has **complete preferences** if they can compare any two objects. An agent has **transitive preferences** if their preferences are internally consistent. While it is natural to think about preferences, it is often more convenient to associate different numbers to different goods, and have the agent choose the good with the highest number. These numbers are called **utilities**. In turn, a utility function tells us the utility associated with each good $x \in X$, and is denoted by $u(x) \in \mathbb{R}$.

Definition 2.3 (Utility Function). *We say a **utility function** $u : G \to \mathbb{R}$ represents an agent's preferences if for all $x, y \in G$, $u(x) \geq u(y)$ if and only if $x \succcurlyeq y$, that is:*

$$\forall x, y \in G, \qquad u(x) \geq u(y) \iff x \succcurlyeq y \tag{1}$$

This means than when an agent has a choice that she prefers to all others according to her preference relation $\succcurlyeq$, if $\succcurlyeq$ can be represented by a utility function $u : G \to \mathbb{R}$ then that choice should maximize u. That is, let $S \subset G$ be a compact set[1]:

$$\forall y \in S \subset G, \;\; x \succcurlyeq y \iff x \in \arg\max_{z \in S} u(z) \tag{2}$$

Theorem 2.4 (Utility Representation Theorem for Finite Choice Sets). *Suppose the agent's preferences, $\succcurlyeq$, are complete and transitive, and that the choice set G is finite. Then there exists a utility function $u : G \to \mathbb{R}$ which represents $\succcurlyeq$*

Proof. For any good x, let $\mathrm{NBT}(x) = \{y \in X | x \succcurlyeq y\}$ be the goods that are "no better than" x. The utility of x is simply given by the number of items in $\mathrm{NBT}(x)$. That is

$$u(x) = |\mathrm{NBT}(x)| \tag{3}$$

We now verify that the construction we have given is valid. Suppose $x \succcurlyeq y$. Pick any $z \in \mathrm{NBT}(y)$ by the definition of $\mathrm{NBT}(y)$, we have $y \succcurlyeq z$ since preferences are complete, we know that z is comparable to x. Transitivity then tells us that $x \succcurlyeq z$, so $z \in \mathrm{NBT}(x)$. We have therefore shown that every element of $\mathrm{NBT}(y)$ is also an element of $\mathrm{NBT}(x)$, that is, $\mathrm{NBT}(y) \subseteq \mathrm{NBT}(x)$. As a result,

$$u(x) = |\mathrm{NBT}(x)| \geq |\mathrm{NBT}(y)| = u(y) \tag{4}$$

which confirms our claim. □

[1]Note that compactness of the choice set S is a sufficient condition for the existence of an element of S that maximizes $\succcurlyeq$

Things get more complicated when the choice set of the agents G becomes infinite, e.g., if $G = \mathbb{R}$ or $G = \mathbb{R}^n$. In this case, the completeness and transitivity axioms are not enough for utility functions to represent a preference relation. For this reason we have to introduce the continuity axiom.

Axiom 2.5 (Continuity axiom)**.** *A preference relation $\succcurlyeq$ on G is **continuous** if for any sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ such that $x_n \succcurlyeq y_n$ and $(x_n, y_n) \to (x, y)$, we have $x \succcurlyeq y$.*

Adding the continuity axiom to our two previous axioms, we can obtain a result similar to theorem 2.4 for infinite choice sets.

Theorem 2.6 (Utility Representation Theorem for Infinite Choice Sets)**.** *Suppose the agent's preferences, $\succcurlyeq$, are complete, transitive, and continuous, and that the choice set G is infinite. Then there exists a continuous utility function $u : G \to \mathbb{R}$ which represents $\succcurlyeq$.*

Theorem 2.7 (Preference Invariance under Monotone Transformations)**.** *Suppose $u : G \to \mathbb{R}$ represents the agent's preferences, $\succcurlyeq$, and $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function. Then the new utility function $v(x) = f(u(x))$ also represents the agent's preferences $\succcurlyeq$.*

Proof: The proof is simply a rewriting of definitions. Suppose $u(x)$ represents the agent's preferences. If $x \succcurlyeq y$ then $u(x) \geq u(y)$ and $f(u(x)) \geq f(u(y))$, so that $v(x) \geq v(y)$. Conversely, if $v(x) \geq v(y)$ then, since $f(.)$ is strictly increasing, $u(x) \geq u(y)$ and $x \succcurlyeq y$. Hence $v(x) \geq v(y) \quad \text{if and only if} \quad x \succcurlyeq y$ and $v(x)$ represents $\succcurlyeq$.

2.2 Properties of Preferences

From now on, we will assume that the choice set G is $\mathbb{R}^m_+$ where m is the number of goods in the market. A **bundle** of goods is simply a vector $\boldsymbol{x} \in \mathbb{R}^m_+$ where $x_i \geq 0$ refers to amount of good i chosen by the agent.

Definition 2.8 (Monotonicity)**.** *Preferences are **monotone** if for any two bundles $\boldsymbol{x} \in \mathbb{R}^m$ and $\boldsymbol{y} \in \mathbb{R}^m$*

$$x_i \geq y_i \textit{ for each } i \tag{5}$$
$$x_i > y_i \textit{ for some } i \tag{6}$$

implies $\boldsymbol{x} \succ \boldsymbol{y}$

In words, preferences are monotone if more of any good makes the agent strictly better off. While monotonicity is stated in terms of preferences, we can rewrite it in terms of utilities.

A preference relation $\succcurlyeq$ represented by a utility function $u : \mathbb{R}^m \to \mathbb{R}$ is **monotone** if for any two bundles $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$:

$$x_i \geq y_i \text{ for each } i \tag{7}$$
$$x_i > y_i \text{ for some } i \tag{8}$$

implies $u(\boldsymbol{x}) > u(\boldsymbol{y})$.

Definition 2.9 (Non-Satiation)**.** *A preference relation is* ***non-satiated****:*

$$\forall \boldsymbol{x} \in \mathbb{R}^m, \;\; \epsilon > 0, \;\; \exists \boldsymbol{y} \in \mathcal{B}_\epsilon(\boldsymbol{x}), \quad \boldsymbol{y} \succ \boldsymbol{x} \tag{9}$$

where $\mathcal{B}_\epsilon(\boldsymbol{x})$ is the ball of radius ϵ centered at $\boldsymbol{x}$.

In words, for any bundle of goods, there exists an arbitrarily close bundle of goods that is preferred. While non-satiation is defined in terms of preferences, we can rewrite it in terms of utilities. A preference relation $\succcurlyeq$ represented by a utility function $u : \mathbb{R}^m \to \mathbb{R}$ is non-satiated if :

$$\forall \boldsymbol{x} \in \mathbb{R}^m, \;\; \epsilon > 0, \;\; \exists \boldsymbol{y} \in \mathcal{B}_\epsilon(\boldsymbol{x}), \quad u(\boldsymbol{y}) > u(\boldsymbol{x}) \tag{10}$$

Convexity Preferences are convex if whenever $x \succcurlyeq y$ then

$$\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} \succcurlyeq y \quad \text{ for all } \lambda \in [0,1] \tag{11}$$

Convexity means that the agent prefers balanced bundles of goods to extreme bundles: if the agent is indifferent between $\boldsymbol{x}$ and $\boldsymbol{y}$ then they prefers the average $\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}$ to either just $\boldsymbol{x}$ or $\boldsymbol{y}$. We can write this assumption in terms of utility functions. Preferences are convex if:

$$u(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \min\{u(\boldsymbol{x}), u(\boldsymbol{y})\} \quad \text{ for all } \lambda \in [0,1] \tag{12}$$

Slightly confusingly, a utility function that satisfies (14) is called **quasi-concave**. Note that any concave function is also quasi-concave.

Definition 2.10 (Convexity)**.** *Preferences are* ***convex*** *if whenever $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ then*

$$\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} \succcurlyeq \boldsymbol{y} \quad \textit{ for all } \lambda \in [0,1] \tag{13}$$

Convexity captures the idea that the agent likes diverse bundles of goods We can write this assumption in terms of utility functions. A preference relation $\succcurlyeq$ represented by a utility function $u : \mathbb{R}^m \to \mathbb{R}$ is **convex** if:

$$u(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \min\{u(\boldsymbol{x}), u(\boldsymbol{y})\} \quad \text{ for all } \lambda \in [0,1] \tag{14}$$

Slightly confusingly, a utility function that satisfies (14) is called **quasi-concave**. Note that any convex function is also quasi-convex.

2.3 Properties of utility functions

We quickly review some important properties of utility functions. Recall that a set U is open if for all $x \in U$ there exists an $\epsilon > 0$ such that the open ball $B_\epsilon(x)$ centered at x with radius ϵ is a subset of U, i.e., $B_\epsilon(x) \subset U$.

Definition 2.11 (Continuity)**.** *A function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be* ***continuous*** *if $f^{-1}(U)$ is open for every $U \subset \mathbb{R}$.*

The definition of continuity we give a topological one but it is equivalent to the $\epsilon - \delta$ definition you might be familiar with.

Definition 2.12 (Quasi-concave)**.** *A function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be concave if*

$$\forall \lambda \in (0,1), \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m, \qquad f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \min\{f(\boldsymbol{x}) + f(\boldsymbol{y})\} \tag{15}$$

If the inequality holds strictly, the f is **strictly quasi-concave**. A utility function is (strictly) quasi-concave iff it represents (strictly) convex preferences.

Definition 2.13 (Concave)**.** *A function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be concave if*

$$\forall \lambda \in (0,1), \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m, \qquad f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y}) \tag{16}$$

If the inequality holds strictly, the f is **strictly concave**.

Note that any concave function is also quasi-concave.

Definition 2.14 (Monotone)**.** *A function $f : \mathbb{R}^m \to \mathbb{R}$ is **monotone** if:*

$$\forall x \geq y, \qquad f(x) \geq f(y) \tag{17}$$

*If the inequality holds strictly then f is **strictly monotone**. Note that a utility function is monotone iff it represents monotone preferences.*

Definition 2.15 (Homogeneous)**.** *A function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be **homogeneous of degree** k if*

$$\forall \boldsymbol{x} \in \mathbb{R}^m, \lambda > 0, f(\lambda \boldsymbol{x}) = \lambda^k f(\boldsymbol{x}) \tag{18}$$

Preferences represented by any homogeneous utility function are called **homothetic preferences**.

3 Consumer Theory Basics

Consumer theory is concerned with how **rational** agents, i.e., agents that behave according to their preferences, make consumption decisions. We call the bundle of goods that a consumer decides to buy a consumer's **demand**. In consumer theory, the demand of buyers can be determined by studying two dual problems, the **utility maximization problem (UMP)** and **the expenditure minimization problem (EMP)**. The UMP refers to the buyer's problem of maximizing its utility constrained by its budget in order to obtain its optimal demand, while the EMP refers to the buyer's problem of minimizing its expenditure constrained by its desired utility level (i.e., optimizing its expenditure function) in order to obtain its optimal demand.

Going forward we will mostly focus on agent who have complete, transitive, continuous, locally non-satiated, and convex preferences. Note that such preferences can be represented with continuous and quasi-concave utility functions that satisfy non-satiation. We will denote the space of utility functions that are continuous, quasi-concave and that represent non-satiated preferences on an infinite choice set G by $\mathcal{U}(G)$. With this definition in mind we first introduce the UMP.

3.1 Utility Maximization Problem

Definition 3.1 (Utility Maximization Problem (UMP))**.** *Let $(u_i, b_i) \in \mathcal{U}(\mathbb{R}^m) \times \mathbb{R}_+$ be a consumer and $\boldsymbol{p} \in \mathbb{R}^m$ be the prices of goods. The **utility maximization problem (UMP)** of (u_i, b_i) is defined as the following optimization problem:*

$$\max_{\boldsymbol{x}:\boldsymbol{x}\cdot\boldsymbol{p}\leq b_i} u_i(\boldsymbol{x}) \tag{19}$$

That is, the UMP, refers to a consumer's problem of buying a utility maximizing bundle of goods when it faces a budget constraint. We can decompose the UMP into two distinct components, 1) the optimal utility, and 2) the the utility maximizing bundle of good.

3.1.1 Indirect Utility and Marshallian Demand

Definition 3.2 (Indirect Utility Function)**.** *The **indirect utility function** $v_i : \mathbb{R}^m_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ takes as input prices $\boldsymbol{p}$ and a budget b_i and outputs the maximum utility i buyer can achieve at those prices and budget, i.e., $v_i(\boldsymbol{p}, b_i) = \max_{\boldsymbol{x}:\boldsymbol{p}\cdot\boldsymbol{x}\leq b_i} u_i(\boldsymbol{x})$.*

If the utility function u_i is continuous, then the indirect utility function is continuous and homogeneous of degree 0 in $\boldsymbol{p}$ and b_i jointly, i.e., $\forall \lambda > 0, v_i(\lambda\boldsymbol{p}, \lambda\boldsymbol{b}) = v_i(\boldsymbol{p}, \boldsymbol{b})$, non-increasing in $\boldsymbol{p}$, strictly increasing in b_i, and convex in $\boldsymbol{p}$ and b_i.

Definition 3.3 (Marshallian Demand)**.** *The **Marshallian demand** is a correspondence $d_i : \mathbb{R}^m_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^m_+$ that takes as input prices $\boldsymbol{p}$ and a budget b_i and outputs the utility maximizing allocation of goods for buyer i, i.e., $d_i(\boldsymbol{p}, b_i) = \arg\max_{\boldsymbol{x}_i:\boldsymbol{p}\cdot\boldsymbol{x}_i\leq b_i} u_i(\boldsymbol{x}_i)$.*

The Marshallian demand is convex-valued if the utility function u_i is continuous and concave, and unique if the utility function is continuous and strictly concave.

3.1.2 Optimality condition for the UMP

We will now derive the **first order optimality condition**[2] for the utility maximization problem. That is, let $u_i \in \mathcal{U}(\mathbb{R})$ be a continuous and concave utility functions, $b_i > 0$, $\boldsymbol{p} \in \mathbb{R}^m$ and $\boldsymbol{x}^* \in \mathbb{R}^m_+$. What (first order) conditions should $\boldsymbol{x}^*$ satisfy to be the utility maximizing demand for the consumer (u_i, b_i) at prices $\boldsymbol{p}$. We first state the theorem:

Theorem 3.4 (Equimarginal Principle)**.** *Consider the utility maximization problem for a consumer* $(u_i, b_i) \in \mathcal{U}(\mathbb{R}) \times \mathbb{R}$ *at prices* $\boldsymbol{p} \in \mathbb{R}^m$*:*

$$\max_{\boldsymbol{x} \in \mathbb{R}^m_+ : \boldsymbol{x} \cdot \boldsymbol{p} \leq b_i} u_i(\boldsymbol{x}) \tag{20}$$

Then, $\boldsymbol{x}^*$ *is a Marshallian demand for consumer* (u_i, b_i) *at prices* $\boldsymbol{p}$ *iff there exists* $\lambda^* \geq 0$ *such that:*

$$x_j^* > 0 \implies \lambda^* = \frac{\left[\frac{\partial u_i}{\partial x_j}\right]_{x_j = x_j^*}}{p_j} \qquad \forall j \in [m] \tag{21}$$

Proof. Consider the utility maximization problem for a consumer (u_i, b_i):

$$\max_{\boldsymbol{x} \in \mathbb{R}^m_+ : \boldsymbol{x} \cdot \boldsymbol{p} \leq b_i} u_i(\boldsymbol{x}) \tag{22}$$

Let $\lambda \in \mathbb{R}$ and $\boldsymbol{\mu} \in \mathbb{R}^m$ be the slack variables associated with the budget constraint and non-negativity constraints respectively. The Lagrangian corresponding to this optimization problem is given by:

$$L(\boldsymbol{x}, \lambda, \boldsymbol{\mu}) = -u_i(\boldsymbol{x}) + \lambda(\boldsymbol{x} \cdot \boldsymbol{p} - b_i) - \boldsymbol{\mu}^T \boldsymbol{x} \tag{23}$$

Let $(\boldsymbol{x}^*, \lambda^*, \boldsymbol{\mu}^*)$ be a saddle point of the above Lagrangian, i.e., $\boldsymbol{x}^* \in \min_{\lambda, \boldsymbol{\mu} \geq 0} L(\boldsymbol{x}, \lambda, \boldsymbol{\mu})$ and $(\lambda^*, \boldsymbol{\mu}^*) \in \max_{\boldsymbol{x} \in \mathbb{R}^m} L(\boldsymbol{x}, \lambda, \boldsymbol{\mu})$. The first order optimality conditions of this problem are :

$$\frac{\partial L}{\partial x_j} = -\left[\frac{\partial u_i}{\partial x_j}\right]_{x_j = x_j^*} + \lambda^* p_j - \mu_j := 0 \tag{24}$$

$$\lambda^* = \frac{\left[\frac{\partial u_i}{\partial x_j}\right]_{x_j = x_j^*} + \mu_j}{p_j} \tag{25}$$

Additionaly, by the KKT complementarity condition, we know that if $x_j > 0$, then $\mu_j = 0$. As a result, the optimality condition for an allocation to be utility maximizing constrained by the budget of the consumer can be equivalently stated as:

$$x_j > 0 \Rightarrow \exists \lambda^*, \text{ such that } \lambda^* = \frac{\left[\frac{\partial u_i}{\partial x_j}\right]_{x_j = x_j^*}}{p_j} \tag{26}$$

□

[2] i.e., the condition the derivative of the objective function needs to satisfy for the a solution to the problem instance to be optimal

In other words, an allocation $\boldsymbol{x}^*$ is optimal if there exists a constant $\lambda^* \geq 0$, such that for any two goods $j, k \in [m]$ for which $x_j^* > 0$ and $x_k^* > 0$, we have:

$$\frac{\left[\frac{\partial u_i}{\partial x_j}\right]_{x_j = x_j^*}}{p_j} = \frac{\left[\frac{\partial u_i}{\partial x_j}\right]_{x_k = x_k^*}}{p_k} \tag{27}$$

That is, an allocation is optimal if it respects the equimarginal principle! Additionally, the Lagrangian λ corresponds to the derivative of the indirect utility function with respect to the budget of the buyer. This quantity is "the shadow price of wealth" or the marginal utility of income, i.e. $\lambda = \frac{\partial}{\partial b_i} v_i(\boldsymbol{p}, b_i)$.

3.2 Expenditure Minimization Problem

A problem that is dual, i.e., parallel but different, to the UMP is the EMP defined as follows:

Definition 3.5 (Expenditure Minimization Problem (EMP)). *Let $(u_i, \nu_i) \in \mathcal{U}(\mathbb{R}^m) \times \mathbb{R}_+$ be a consumer. The **expenditure minimization problem** is defined as follows:*

$$\min_{\boldsymbol{x} : u_i(\boldsymbol{x}) \geq \nu_i} \boldsymbol{p} \cdot \boldsymbol{x} \tag{28}$$

That is, the expenditure minimization problem refers to a consumer's problem of minimizing their expenditure while desiring a minimum utility level they would like to achieve.

3.2.1 Expenditure Function and Hicksian Demand

We can decompose the EMP into two distinct components, 1) the minimum spending that achieves the desired utility level, and 2) the bundle of goods that minimizes the expenditure of the buyer while achieving the desired utility level.

Definition 3.6 (Expenditure Function). *The **expenditure function** $e_i : \mathbb{R}_+^m \times \mathbb{R}_+ \to \mathbb{R}_+$ takes as input prices $\boldsymbol{p}$ and a utility level ν and returns the minimum cost required to achieve the utility level ν at given prices $\boldsymbol{p}$, i.e., $e_i(\boldsymbol{p}, \nu_i) = \min_{\boldsymbol{x}_i : u_i(\boldsymbol{x}_i) \geq \nu_i} \boldsymbol{p} \cdot \boldsymbol{x}_i$.*

If the utility function u_i is continuous, then the expenditure function is continuous and homogeneous of degree 1 in $\boldsymbol{p}$ and ν_i jointly, non-decreasing in $\boldsymbol{p}$, strictly increasing in ν_i, and concave in $\boldsymbol{p}$.

Definition 3.7 (Hicksian Demand). *The **Hicksian demand** is a correspondence $h_i : \mathbb{R}_+^m \times \mathbb{R}_+ \rightrightarrows \mathbb{R}_+$ that takes as input prices $\boldsymbol{p}$ and a utility level ν_i and outputs the cost-minimizing allocation of goods at utility level ν_i, i.e., $h_i(\boldsymbol{p}, \nu_i) = \arg\min_{\boldsymbol{x}_i : u_i(\boldsymbol{x}_i) \geq \nu_i} \boldsymbol{p} \cdot \boldsymbol{x}_i$.*

If the utility function u_i is continuous, then the Hicksian demand is homogeneous of degree 1 in $\boldsymbol{p}$, and is a convex set which is unique iff the the utilty function u_i of the buyer is strictly concave [14, 15].

3.2.2 Optimality condition for the EMP

We now state the optimality condition for the expenditure minimization problem. We skip the proof as it is similar to that of theorem 3.4.

Theorem 3.8 (Equimarginal Principle for the EMP). *Consider the expenditure minimization problem for a consumer $(u_i, \nu_i) \in \mathcal{U}(\mathbb{R}) \times \mathbb{R}$ at prices $\boldsymbol{p} \in \mathbb{R}^m$:*

$$\min_{\boldsymbol{x} \in \mathbb{R}^m_+ : u_i(\boldsymbol{x}) \geq \nu_i} \boldsymbol{p} \cdot \boldsymbol{x} \tag{29}$$

Then, $\boldsymbol{x}^* \in \mathbb{R}^m$ is a Hicksian demand for consumer (u_i, ν_i) at prices $\boldsymbol{p}$ iff there exists $\lambda^* \geq 0$ such that:

$$x_j^* > 0 \implies \lambda^* = \frac{p_j}{\left[\frac{\partial u_i}{\partial x_j}\right]_{x_j = x_j^*}} \forall j \in [m] \tag{30}$$

The UMP and EMP are related in many ways whether if it is through convex conjugacy duality or via economic duality. We refer the reader to [4, 14, 15] for a more in-depth discussion of these duality relationships and only note the following relationship between the UMP and EMP:

$$\forall b_i \in \mathbb{R}_+ \qquad e_i(\boldsymbol{p}, v_i(\boldsymbol{p}, b_i)) = b_i \tag{31}$$

$$\forall \nu_i \in \mathbb{R}_+ \qquad v_i(\boldsymbol{p}, e_i(\boldsymbol{p}, \nu_i)) = \nu_i \tag{32}$$

$$\forall b_i \in \mathbb{R}_+ \qquad h_i(\boldsymbol{p}, v_i(\boldsymbol{p}, b_i)) = d_i(\boldsymbol{p}, b_i) \tag{33}$$

$$\forall \nu_i \in \mathbb{R}_+ \qquad d_i(\boldsymbol{p}, e_i(\boldsymbol{p}, \nu_i)) = h_i(\boldsymbol{p}, \nu_i) \tag{34}$$

A good $j \in [m]$ is said to be a **gross substitute (resp. complement)** for a good $k \in [m]/\{j\}$ if $\sum_{i \in [n]} d_{ij}(\boldsymbol{p}, b_i)$ is increasing (resp. decreasing) in p_k. If the aggregate demand, $\sum_{i \in [n]} d_{ij}(\boldsymbol{p}, b_i)$, for good k is instead weakly increasing (resp. weakly decreasing), good j is said to be a **weak gross substitute** (resp. **weak gross complement**) for good k.

3.3 Important Classes of Utility Functions

In this section, we introduce a few important classes of utility functions which are of computational interest. Let m be the number of goods. Let $\boldsymbol{x} \in \mathbb{R}^m$ be a bundle of goods in the space of goods. Let $u : \mathbb{R}^m \to \mathbb{R}$ be the utility function of the agent and let $\boldsymbol{v} \in \mathbb{R}^m$ be a vector of preference parameters for each good $j \in [m]$. We define a few important classes of utility functions.

Definition 3.9 (Linear Utility). ***Linear Utilities*** *represent preferences over goods that are* ***perfect substitutes****. That is, having more of a good leads the consumer to want less of other goods. An example of this is sugar and artificial sweeteners. If a buyer buys sugar, then it will not use artificial sweeteners. Mathematically, linear utilities are defined as:*

$$u(\boldsymbol{x}) = \sum_{j=1}^{m} v_j x_j \tag{35}$$

Note that linear utilities are affine and not strictly convex. As a result, the Marshallian demand and Hicksian demand is not unique for linear utilities. This means that we do not have closed form solutions for those quantities, however, since the indirect utility and expenditure function's output are singleton valued, we have closed form solutions for those:

Theorem 3.10 (Linear Utility). *Let $u : \mathbb{R}^m \to \mathbb{R}$ be of the form of definition 3.9. Then the following holds:*

$$v_i(\boldsymbol{p}, b) = \max_{j \in [m]} \left(\frac{v_j}{p_j} \right) b \tag{36}$$

$$e_i(\boldsymbol{p}, \nu) = \min_{j \in [m]} \left(\frac{p_j}{\boldsymbol{v}_j} \right) \nu \tag{37}$$

Definition 3.11 (Leontief Utility). ***Leontief Utilities*** *represent preferences over goods that are* ***perfect complements****. That is in order to derive utility from one good, the buyer also needs to have more of other goods. An example is that a buyer needs both the left and right pair of a shoe to derive utility from the shoes. It cannot derive any utility from only the left or right shoe. Mathematically, Leontief utilities are defined as:*

$$u(\boldsymbol{x}) = \min_{j=1,\ldots,m} \left\{ \frac{x_j}{v_j} \right\} \tag{38}$$

For Leontief utilities, we have convenient closed form solutions for the UMP and EMP:

Theorem 3.12. *Let $u : \mathbb{R}^m \to \mathbb{R}$ be of the form of definition 3.11. Then the following holds:*

$$v(\boldsymbol{p}, b) = \frac{b}{\sum_{j \in [m]} p_j v_j} \tag{39}$$

$$d_j(\boldsymbol{p}, b) = \frac{v_j b}{\sum_{j \in [m]} p_j v_j} \qquad \forall j \in [m] \tag{40}$$

$$e(\boldsymbol{p}, \nu) = \nu \sum_{j \in [m]} p_j v_j \tag{41}$$

$$h_j(\boldsymbol{p}, \nu) = \nu v_j \qquad \forall j \in [m] \tag{42}$$

Definition 3.13 (Cobb-Douglas Utility). ***Cobb-Douglas Utilities*** *represent preferences over goods that in-between between perfect substitutes and perfect complements. That is, under Cobb-Douglas utilities a buyer prefers bundle of goods that are balanced to bundles of goods that are extreme. Mathematically, Cobb-Douglas utilities are defined as:*

$$u(\boldsymbol{x}) = \prod_{j=1}^{m} x_j^{v_j} \tag{43}$$

where $\sum_{j=1}^{m} v_j = 1$.

For Cobb-Douglas utilities, we have convenient closed form solutions for the UMP and EMP:

Theorem 3.14. *Let $u : \mathbb{R}^m \to \mathbb{R}$ be of the form of definition 3.13. Then the following holds:*

$$v(\boldsymbol{p}, b) = b \prod_{j=1}^{m} \left(\frac{v_j}{p_j}\right)^{v_j} \tag{44}$$

$$d_j(\boldsymbol{p}, b) = \frac{v_j b}{p_j} \qquad \forall j \in [m] \tag{45}$$

$$e(\boldsymbol{p}, \nu) = \nu \prod_{j=1}^{m} \left(\frac{p_j}{v_j}\right)^{v_j} \tag{46}$$

$$h_j(\boldsymbol{p}, \nu) = \nu \left(\frac{v_j}{p_j}\right) \prod_{j=1}^{m} \left(\frac{p_j}{v_j}\right)^{v_j} \qquad \forall j \in [m] \tag{47}$$

These three utility functions happen to be special cases of the **constant elasticity of substitution (CES) utility**.

Definition 3.15 (Constant Elasticity of Substitution (CES) Utility)**.** ***Constant Elasticity of Substitution (CES)*** *utilities represent preferences over goods for which the rate at which the consumer can change its consumption of any two goods while staying on the same indifference curve is constant and is defined as:*

$$u(\boldsymbol{x}) = \left(\sum_{i=1}^{n} v_i x_i^{\rho}\right)^{\frac{1}{\rho}} \tag{48}$$

where $-\infty < \rho \leq 1$ *and* $\rho \neq 0$*.*

1. When $\rho = 1$ then the CES utility function is exactly the linear utility function
2. When $\rho \to -\infty$, the CES utility function is exactly the Leontief utility function.
3. When $\rho \to 0$, the CES utility function is exactly the Cobb-Douglas utility function

To see the above for the limits, one can use l'hopital's rule to calculate the limit of $\log(u(\boldsymbol{x}))$ at the desired limit value and from there one can deduce the value of $u(\boldsymbol{x})$ at the desired limit.[3] For $0 < \rho \leq 1$, goods are weak gross substitutes, for $\rho = 1$, goods are perfect substitutes, and for $\rho < 0$, goods are complementary. For $\rho < 1$, the CES utility function is **strictly** concave while it is concave for $\rho \leq 1$. To see this you can take the second derivative of the utility function. Additionally, CES utilities are homogeneous of degree 1 $u(\lambda \boldsymbol{x}) = (\sum_{i=1}^{n} v_i \lambda^{\rho} x_i^{\rho})^{\frac{1}{\rho}} = (\lambda^{\rho})^{\frac{1}{\rho}} (\sum_{i=1}^{n} v_i x_i^{\rho})^{\frac{1}{\rho}} = \lambda (\sum_{i=1}^{n} v_i x_i^{\rho})^{\frac{1}{\rho}}$. The **elasticity of substitution** of CES utility functions is given by $\frac{1}{1-\rho}$.

For CES utilities with $\rho < 1$, we have convenient closed form solutions for the UMP and EMP:

Theorem 3.16. *Let* $u : \mathbb{R}^m \to \mathbb{R}$ *be of the form of definition 3.15. Then the following holds:*

[3] More on the limit calculation here

$$v(\boldsymbol{p}, b) = b\left(\sum_{k=1}^{n} v_k^{\frac{1}{1-\rho}} p_k^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} \tag{49}$$

$$d_j(\boldsymbol{p}, b) = b\frac{v_i^{\frac{1}{1-\rho}} p_i^{\frac{1}{\rho-1}}}{\sum_{k=1}^{n} v_k^{\frac{1}{1-\rho}} p_k^{\frac{\rho}{\rho-1}}} \qquad \forall j \in [m] \tag{50}$$

$$e(\boldsymbol{p}, \nu) = \nu\left(\sum_{k=1}^{n} v_k^{\frac{1}{1-\rho}} p_k^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} \tag{51}$$

$$h_j(\boldsymbol{p}, \nu) = \nu\left(\sum_{k=1}^{n} v_k^{\frac{1}{1-\rho}} p_k^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} \frac{v_i^{\frac{1}{1-\rho}} p_i^{\frac{1}{\rho-1}}}{\sum_{k=1}^{n} v_k^{\frac{1}{1-\rho}} p_k^{\frac{\rho}{\rho-1}}} \qquad \forall j \in [m] \tag{52}$$

Definition 3.17 (Quasilinear Utility)**.** *A* ***quasilinear utility*** *is a utility function of the following form:*

$$u\left(\boldsymbol{x}\right) = \alpha x_1 + \theta\left(x_2, \ldots, x_n\right) \tag{53}$$

where θ is an arbitrary function and $\alpha \geq 0$. Unless otherwise noted, when talking about quasilinear utilities in the literature, we refer utility functions of the following form which conforms with the general quasilinear form:

$$u\left(\boldsymbol{x}; \boldsymbol{p}\right) = \sum_{1=1}^{n} x_j\left(v_j - p_j\right) \tag{54}$$

where p_j is the price of the j^{th} good.

Note that quasilinear utilities are affine and not strictly convex. As a result, the Marshallian demand and Hicksian demand is not unique for quasilinear utilities. This means that we do not have closed form solutions for those quantities, however, since the indirect utility and expenditure function's output are singleton valued, we have closed form solutions for those:

Theorem 3.18 (Quasilinear Utility)**.** *Let $u : \mathbb{R}^m \to \mathbb{R}$ be of the form of eq.* (54)*. Then the following holds:*

$$v_i(\boldsymbol{p}, b) = \max\left\{1, \max_{j\in[m]}\left(\frac{v_j}{p_j}\right)\right\} b \tag{55}$$

$$e_i(\boldsymbol{p}, \nu) = \min_{j\in[m]}\left(\frac{p_j}{\boldsymbol{v}_j}\right) \nu \tag{56}$$

Definition 3.19 (Stone-Geary Utility)**.** ***Stone-Geary Utilities*** *represent preferences over goods that in-between between perfect substitutes and perfect complements and that are not homothetic. That is, under Stone-Geary utilities a buyer prefers a fixed bundle up to a subsistence level, and after that level the buyer prefers bundle of goods that are balanced to bundles of goods that are extreme. Mathematically, Stone-Geary utilities are defined as:*

$$u(\boldsymbol{x}) = \prod_{j=1}^{m} \left(x_j - w_j\right)^{v_j} \tag{57}$$

where $\sum_{j=1}^{m} v_j = 1$.

Theorem 3.20 (Stone-Geary Utility)**.** *Let* $u : \mathbb{R}^m \to \mathbb{R}$ *be of the form of definition 3.19. Then the following holds:*

$$v(\boldsymbol{p}, b) = \prod_{j=1}^{m} \left(\frac{v_j \left(b - \sum_{j=1}^{m} w_k p_k \right)}{p_j} \right)^{v_j} \tag{58}$$

$$d_j(\boldsymbol{p}, b) = w_j + v_j \frac{b_i - \sum_{j=1}^{m} w_j p_j}{p_j} \qquad \forall j \in [m] \tag{59}$$

$$e(\boldsymbol{p}, \nu) = h_j(\boldsymbol{p}, \nu) \qquad = \tag{60}$$

4 Fisher Market Model

The Fisher market model is the first model of a market that proves that a competitive equilibrium exists in a market, that is, there exists prices and allocation of goods that maximize the utility of buyers and clears the market. In other words, the fisher market proves that in a market prices exist such that the supply is equal to demand. It was introduce by Irving Fisher in the 1870s who provided a proof of equilibrium existence via Hydraulic Machine [5]!

4.1 Model

A **Fisher market** consists of:

1. Finite set of n heterogeneous **buyers**.
2. Finite set of m heterogeneous **goods**.
3. Each buyer $i \in [n]$ is characterized by:
   (a) a budget $b_i \in \mathbb{R}$.
   (b) a utility function, $u_i : \mathbb{R}^m_+ \to \mathbb{R}$, giving the utility that buyer i derives from each bundle of goods.
4. WLOG, we assume that there is only one unit of each good $j \in [m]$. The results/solutions we provide in this section can be applied to the more general settings in which the number of copies of each good is different than 1.

A Fisher market is a tuple $(n, m, U, \boldsymbol{b})$. When clear from context, we simply denote $(U, \boldsymbol{b})$.

4.2 Model Outcome

An **allocation** $\boldsymbol{X} \in \mathbb{R}^{n\times m}_+$ is a map from goods to buyers, represented as a matrix, s.t. $x_{ij} \geq 0$ denotes the amount of good $j \in [m]$ allocated to buyer $i \in [n]$. When utility function U belong to the same class, we refer to the Fisher market by the class of utility functions, e.g., if U is a set of linear utility functions $(U, \boldsymbol{b})$ is a linear Fisher market.

Definition 4.1 (Feasible allocation)**.** *An allocation is said to be* ***feasible*** *if no more than 1 unit of a good $j \in [m]$ is allocated in total, across all buyers,* $\sum\limits_{i\in[n]} x_{ij} \leq 1$

Goods are assigned **prices** $\boldsymbol{p} \in \mathbb{R}^m_+$. Note that prices are **anonymous**, in that all copies of good j are assigned the same price p_j. An **outcome** is a pair $(\boldsymbol{X}, \boldsymbol{p}) \in \mathbb{R}^{n\times m}_+ \times \mathbb{R}^m_+$ consisting of an allocation and prices respectively. For any outcome $(\boldsymbol{X}, \boldsymbol{p})$, we define the **demand** for good $j \in [m]$ as $\sum\limits_{i=1}^{n} x_{ij}$ and the **supply** of good j as 1 (since we assumed that there is 1 unit of each good).

Definition 4.2 (Utility Maximization)**.** *An outcome $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ is* ***utility maximizing*** *if no buyer would prefer a different feasible allocation of goods than theirs, at the outcome's prices, that is:*

$$\forall i \in [n], \boldsymbol{x}^*_i, \in \arg\max_{\boldsymbol{x}:\boldsymbol{x}\cdot\boldsymbol{p}\leq b_i} u_i(\boldsymbol{x}_i). \tag{61}$$

Definition 4.3 (Market Clearance)**.** *An outcome $(\boldsymbol{X}, \boldsymbol{p})$ is **market clearing** if the demand of any goods for which the price is greater than 0 is equal to its supply and for goods which are priced at 0 its demand is less than or equal to its supply, that is:*

$$p_j > 0 \implies \sum_{i \in [n]} x_{ij} = 1 \text{ and } p_j = 0 \implies \sum_{i \in [n]} x_{ij} \leq 1 \qquad \forall j \in [m] \tag{62}$$

or equivalently

$$\sum_{j \in [m]} p_j (\sum_{i \in [n]} x_{ij} - 1) = 0 \text{ and } \forall j \in [m] \sum_{i \in [n]} x_{ij} \leq 1 \tag{63}$$

The last condition, i.e., $\sum_{j \in [m]} p_j(\sum_{i \in [n]} x_{ij} - 1) = 0$ and $\forall j \in [m], \sum_{i \in [n]} x_{ij} \leq 1$, are respectively called **Walras' law** and the **feasibility conditions**.

Definition 4.4 (Walrasian/Competitive Equilibrium)**.** *An outcome is an **Competitive (or Walrasian) equilibrium** if it is utility maximizing and market clearing.*

The existence of a Walrasian equilibrium for any continuous, concave utility function can be shown through the use of Sperner's lemma[4] (or alternatively by Brouwer's fixed point theorem). Sperner's lemma is a combinatorial analog of Brouwer's fixed point theorem, which posits the existence of a fixed of a function in a very general setting. This proof however is non-constructive in that we cannot use it to compute an equilibrium outcome of the Fisher market. We will provide a convex program to compute an equilibrium for Fisher Markets with **continuous, concave, and homogeneous (CCH)** utility functions. This also is a constructive proof of the existence of an equilibrium for buyers with preferences represented by CCH utility functions since the convex program is guaranteed to have an optimal value. We refer to Fisher markets consisting of buyers with preferences represented by CCH utilities as **homothetic** Fisher markets.

4.3 Computing Fisher Market Walrasian Equilibria

Before introducing the Eisenberg-Gale convex program which provides an equilibrium solution for homothetic Fisher markets, we have to prove the following result due to Euler:

Before completing the proof, we need to prove one more theorem.

Theorem 4.5 (Euler's Theorem)**.** *Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a homogeneous function of degree k that is continuous and differentiable on $\mathbb{R}^n_{>0}$, then the following holds:*

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(\boldsymbol{x}) x_i = k f(\boldsymbol{x}) \tag{64}$$

Proof. Assume that f is a homogeneous function of degree k. Let $\boldsymbol{x} \in \mathbb{R}^n_{>0}$. Define $g : (0, \infty) \to \mathbb{R}$ such that:

$$g(\lambda) = f(\lambda \boldsymbol{x}) - \lambda^k f(\boldsymbol{x}) \tag{65}$$

[4] https://en.wikipedia.org/wiki/Fisher_market#cite_note-3

Due to f being homogeneous, this function has a value of 0 for its entire domain. This implies that its derivative is also 0 for its domain:

$$g'(\lambda) = 0 \tag{66}$$

Using the chain rule, we also know that the derivative of g can also be calculated as:

$$g'(\lambda) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(\boldsymbol{x}) x_i - k f(\boldsymbol{x}) \tag{67}$$

Using (66) and setting $\lambda = 0$, we then get:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(\boldsymbol{x}) x_i = k f(\boldsymbol{x}) \tag{68}$$

□

Theorem 4.6 (Eisenberg-Gale Convex Program). *Let $(n, m, U, \boldsymbol{b})$ be a homothetic Fisher market, i.e., U is a set of CCH utility functions, then the primal of following program computes equilibrium allocations $\boldsymbol{X}^*$:*

Primal

$$\max_{\boldsymbol{X}} \sum_{i=1}^{n} b_i \log \left(u_i \left(\boldsymbol{x}_i \right) \right) \tag{69}$$

$$\forall j \in [m], \qquad \sum_{i=1}^{n} x_{ij} \leq 1 \tag{70}$$

$$\forall i \in [n], j \in [m] \qquad x_{ij} \geq 0 \tag{71}$$

Dual

$$\min_{\boldsymbol{p}} \sum_{j \in [m]} p_j + \sum_{i \in [n]} b_i \log \left(v_i(\boldsymbol{p}, b_i) \right) \tag{72}$$

$$\forall j \in [m] \qquad p_j \geq 0 \tag{73}$$

where v_i is the indirect utility function associated with u_i.

Proof. Without loss of generality, assume that the degree of the utility function is $k = 1$. This does not lose the generality of our result since we can transform any homogeneous function $u(x)$ of degree k, to a homogeneous function of degree 1 using the monotonically non-decreasing transformation $u \mapsto \sqrt[k]{u}$. This allows us to conserve all properties of the utility function, more specificaly the preference relations between goods by theorem 2.7. Firstly, note that since the utilities are concave and the logarithm function is a concave function the objective function is also concave. Furthermore, as the constraints are all affine, the program we propose is feasible and bounded. We then write down the the Lagrangian L for this convex program, using slack variables $\forall j \in [m],\ p_j \geq 0$ and $\forall j \in [m], i \in [n],\ \ \lambda_{ij}$:

$$L(\boldsymbol{X}, \boldsymbol{p}, \boldsymbol{\lambda}) = \sum_{i=1}^{n} -b_i \log(u_i(\boldsymbol{x}_i)) + \sum_{j=1}^{m} p_j \left(\sum_{i=1}^{n} x_{ij} - 1 \right) + \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_{ij} (-x_{ij}) \tag{74}$$

Any optimal solution of a convex program is guaranteed to satisfy a series of conditions called the Karush–Kuhn–Tucker (KKT) conditions.[5]. We will now argue that a saddle point $(\boldsymbol{X}^*, \boldsymbol{p}^*)$[6] of the Lagrangian L constitute a Walrasian equilibrium of the Fisher market $(U, \boldsymbol{b})$. From the complementary slackness condition, we have the following two conditions:

$$\sum_{j \in [m]} p_j^* \left(x_{ij}^* - 1 \right) = 0 \tag{75}$$

$$\sum_{i \in [n]} \sum_{j \in [m]} \lambda_{ij}^* x_{ij}^* = 0 \tag{76}$$

Combining eq. (75) with eq. (100), we get that $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ satisfies market clearance since these two conditions are Walras' law and feasibility respectively, i.e., the definition of market clearance.
From the stationarity conditions, we get:

$$\frac{\partial L}{\partial x_{ij}} = \frac{-b_i}{u_i(\boldsymbol{x}_i^*)} \left[\frac{\partial u_i}{\partial x_{ij}} \right]_{\boldsymbol{x}_i = \boldsymbol{x}_i^*} + p_j^* - \lambda_{ij}^* = 0 \tag{77}$$

We reorganize eq. (77) to prove that consumers do not spend more than their budget:

$$p_j^* = \frac{b_i}{u_i(\boldsymbol{x}_i^*)} \left[\frac{\partial u_i}{\partial x_{ij}} \right]_{\boldsymbol{x}_i = \boldsymbol{x}_i^*} + \lambda_{ij}^* \tag{78}$$

If $x_{ij} > 0$, by eq. (76) we know that $\lambda_{ij} = 0$ which gives us:

$$p_j^* = \frac{b_i}{u_i(\boldsymbol{x}_i^*)} \left[\frac{\partial u_i}{\partial x_{ij}} \right]_{\boldsymbol{x}_i = \boldsymbol{x}_i^*} \tag{79}$$

Multiplying both side by x_{ij}, we obtain:

$$p_j^* x_{ij}^* = \frac{b_i x_{ij}^*}{u_i(\boldsymbol{x}_i^*)} \left[\frac{\partial u_i}{\partial x_{ij}} \right]_{\boldsymbol{x}_i = \boldsymbol{x}_i^*} \tag{80}$$

Summing up both sides across all goods, we obtain:

[5] More background on this can be found here

[6] We ignore the slack variable $\mu\lambda^*$ that normally would be part of the saddle point since they do not play a role in the Fisher market outcome.

$$\sum_{j=1}^{m} p_j^* x_{ij}^* = b_i \sum_{j=1}^{m} \frac{x_{ij}^*}{u_i(\boldsymbol{x}_i^*)} \left[\frac{\partial u_i}{\partial x_{ij}}\right]_{\boldsymbol{x}_i=\boldsymbol{x}_i^*} \tag{81}$$

$$\sum_{j=1}^{m} p_j^* x_{ij}^* = b_i \sum_{j=1}^{m} \frac{x_{ij}^*}{u_i(\boldsymbol{x}_i^*)} \left[\frac{\partial u_i}{\partial x_{ij}}\right]_{\boldsymbol{x}_i=\boldsymbol{x}_i^*} \tag{82}$$

$$\sum_{j=1}^{m} p_j^* x_{ij}^* = \frac{b_i}{u_i(\boldsymbol{x}_i^*)} \sum_{j=1}^{m} \left[\frac{\partial u_i}{\partial x_{ij}}\right]_{\boldsymbol{x}_i=\boldsymbol{x}_i^*} x_{ij}^* \tag{83}$$

$$\tag{84}$$

Using Euler's theorem with $k = 1$ (since by our assumption the utility functions are homogeneous of degree 1) we get:

$$\sum_{j=1}^{m} p_j^* x_{ij}^* = \frac{b_i}{u_i(\boldsymbol{x}_i^*)} \sum_{j=1}^{m} \left[\frac{\partial u_i}{\partial x_{ij}}\right]_{\boldsymbol{x}_i=\boldsymbol{x}_i^*} x_{ij}^* \tag{85}$$

$$\sum_{j=1}^{m} p_j^* x_{ij}^* = \frac{b_i}{u_i(\boldsymbol{x}_i^*)} u_i(\boldsymbol{x}_i^*) \tag{86}$$

$$\sum_{j=1}^{m} p_j^* x_{ij}^* = b_i \tag{87}$$

Notice that the left hand side of this expression is exactly the spending of any buyer $i \in [n]$ at the outcome $(\boldsymbol{X}^*, \boldsymbol{p}^*)$. This result implies that consumers are not spending more than their budget.

We will now show that $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ is utility maximizing. Going back to eq. (77), we have:

$$\frac{\partial L}{\partial x_{ij}} = \frac{-b_i}{u_i(\boldsymbol{x}_i^*)} \left[\frac{\partial u_i}{\partial x_{ij}}\right]_{\boldsymbol{x}_i=\boldsymbol{x}_i^*} + p_j^* - \lambda_{ij}^* = 0 \tag{88}$$

If $x_{ij} > 0$, by eq. (76) we know that $\lambda_{ij} = 0$ which gives us:

$$p_j^* = \frac{b_i}{u_i(\boldsymbol{x}_i^*)} \left[\frac{\partial u_i}{\partial x_{ij}}\right]_{\boldsymbol{x}_i=\boldsymbol{x}_i^*} \tag{89}$$

$$\frac{u_i(\boldsymbol{x}_i^*)}{b_i} = \frac{\left[\frac{\partial u_i}{\partial x_{ij}}\right]_{\boldsymbol{x}_i=\boldsymbol{x}_i^*}}{p_j^*} \tag{90}$$

The last condition is exactly the equimarginal principle, theorem 3.4, hence $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ is utility maximizing.

Furthermore, we can also find the dual of the program we proposed by simply taking the maximum of the Lagrangian over all allocations which gives:

Dual Objective

$$\max_{\boldsymbol{X} \in \mathbb{R}_+^{n \times m}} L(\boldsymbol{X}, \boldsymbol{p}, \lambda) = \sum_{j \in [m]} p_j + \sum_{i \in [n]} b_i \log \left(\max_{\boldsymbol{x}_i \geq 0 : \boldsymbol{x}_i \cdot \boldsymbol{p} \leq b_i} u_i \right) \tag{91}$$

$$= \sum_{j \in [m]} p_j + \sum_{i \in [n]} b_i \log \left(v_i(\boldsymbol{p}, b_i) \right) \tag{92}$$

The feasible set of the dual is simply $\mathbb{R}^m$ which essentially makes the problem unconstrained. □

From the above theorem and using closed form solutions of the indirect utility function for different utility functions, we can obtain the following corollaries. Note that the duals are slightly different than that proposed in theorem 4.6 to be consistent with the way these programs are presented in the literature.

Corollary 4.7 (Linear Fisher Market)**.** *Let $(U, \boldsymbol{b})$ be a linear Fisher market, i.e., U is a set of linear utility functions. Then, the equilibrium of the Fisher market $(U, \boldsymbol{b})$ is given by:*

Primal

$$\max_{\boldsymbol{X}} \sum_{i=1}^{n} b_i \log \left(\sum_{j \in [m]} v_{ij} x_{ij} \right) \tag{93}$$

$$\forall j \in [m], \quad \sum_{i=1}^{n} x_{ij} \leq 1 \tag{94}$$

$$\forall i \in [n], j \in [m] \quad x_{ij} \geq 0 \tag{95}$$

Dual

$$\min_{\boldsymbol{p}} \sum_{j \in [m]} p_j - \sum_{i \in [n]} b_i \log \left(\min_{j \in [m]} \left(\frac{p_j}{v_j} \right) \right) \tag{96}$$

$$\forall j \in [m] \quad p_j \geq 0 \tag{97}$$

Corollary 4.8 (Leontief Fisher Market)**.** *Let $(U, \boldsymbol{b})$ be a Leontief Fisher market, i.e., U is a set of linear utility functions. Then, the equilibrium of the Fisher market $(U, \boldsymbol{b})$ is given by:*

Primal

$$\max_{\boldsymbol{X}} \sum_{i=1}^{n} b_i \log (u_i) \tag{98}$$

$$\forall i \in [n], j \in [m] \quad u_i \leq \frac{x_{ij}}{v_{ij}} \tag{99}$$

$$\forall j \in [m], \quad \sum_{i=1}^{n} x_{ij} \leq 1 \tag{100}$$

$$\forall i \in [n], j \in [m] \quad x_{ij} \geq 0 \tag{101}$$

Dual

$$\min_{\boldsymbol{p}} \sum_{j \in [m]} p_j - \sum_{i \in [n]} b_i \log \left(\sum_{j \in [m]} p_j v_{ij} \right) \tag{102}$$

$$\forall j \in [m] \quad p_j \geq 0 \tag{103}$$

We note that for Cobb-Douglas Fisher markets the equilibrium outcome $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ can be written in closed form.

Theorem 4.9 (Cobb-Douglas Fisher Market Equilibrium). *Let $(U, \boldsymbol{b})$ be a Cobb-Douglas Fisher market, i.e., U is a set of Cobb-Douglas utility functions. Then, the equilibrium of the Fisher market $(U, \boldsymbol{b})$ is given by:*

$$x_{ij}^* = \frac{v_{ij} b_i}{\sum_{k \in [n]} v_{kj} b_k} \qquad \forall j \in [m], i \in [n] \tag{104}$$

$$p_j^* = \sum_{i \in [n]} v_{ij} b_i \qquad \forall j \in [m] \tag{105}$$

There are two proof one by simply solving the Eisenberg-Gale program by hand to obtain a closed form soluton and the other as follows:

Proof. We assume that the valuations of the buyers are normalized such that $\sum_{j \in [m]} v_{ij} = 1$ Recall that the demand of buyer i for good j, $D^i(\boldsymbol{p})$ at given prices $\boldsymbol{p}$ for Cobb-Douglas utilities is:

$$D_j^i(\boldsymbol{p}) = \frac{v_{ij} b_i}{p_j} \tag{106}$$

Summing up both sides across all buyer, we get:

$$\sum_{i \in [n]} D_j^i(\boldsymbol{p}) = \sum_{i \in [n]} \frac{v_{ij} b_i}{p_j} \tag{107}$$

$$\sum_{i \in [n]} D_j^i(\boldsymbol{p}) = \frac{1}{p_j} \sum_{i \in [n]} v_{ij} b_i \tag{108}$$

The left hand side of this expression is the demand for good j. Since in a fisher market there is one unit of each good, we can set the demand equal to 1 and solve for the equilibrium price of good j, p_j^*:

$$1 = \frac{1}{p_j^*} \sum_{i \in [n]} v_{ij} b_i \tag{109}$$

$$p_j^* = \sum_{i \in [n]} v_{ij} b_i \tag{110}$$

$$\tag{111}$$

Finally, by substituting our formula for the equilibrium price of good j into the demand set formula, we get the equilibrium allocation x_{ij}^* of buyer i for good j:

$$x_{ij}^* = \frac{v_{ij} b_i}{\sum_{k \in [n]} v_{kj} b_k} \tag{112}$$

Note that this closed form solutions assumes that the valuations are of the following form, $\forall i \in [n], \; \sum_{j \in [m]} v_{ij} = 1$ and that there is only one unit of each good in the market. □

Although the Eisenberg-Gale program is known to provide solutions for Fisher markets with CCH utility functions, it does not provide a solution for Fisher markets with quasilinear utility functions, even though quasilinear utilities are homogeneous. The reason why the Eisenberg-Gale program fails in this case is because quasilinear utilities are parameterized by prices, which renders the Eisenberg-Gale program's objective function ill-defined.[7]

This difficulty arises because historically, quasilinear utilities (and more broadly all utility functions parameterized by prices) were not seen as well-defined utility functions, since they attribute value to money, which intrinsically has no value, a controversy known as Hahn's problem [13] within mainstream microeconomics. A remedy to this problem was brought forward by Devanur [10], who provided a partial answer by noticing that Fisher markets with quasilinear utilities can be solved via Shmyrev's program [16]. The primal of Shmyrev's program captures the equilibrium allocations $\boldsymbol{X}^*$ while the dual captures the equilibrium prices $\boldsymbol{p}^*$ os any quasilinear Fisher market $(U, \boldsymbol{b})$:

Theorem 4.10 (Devanur's Program)**.** *Let $(U, \boldsymbol{b})$ be a quasilinear Fisher market, i.e., U is a set of functions of the form $u_i(\boldsymbol{x}_i; \boldsymbol{v}_i, \boldsymbol{p}) = \sum_{j \in [m]} (v_{ij} - p_j)$, then the tuple of optimal solutions $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ for the primal and dual respectively, comprise of a Walrasian equilibrium for the Fisher market $(U, \boldsymbol{b})$.*

Primal

$$\max_{\boldsymbol{X}, \boldsymbol{u}, \boldsymbol{v}} \sum_{i=1}^{n} b_i \log(u_i) - v_i \tag{113}$$

$$\forall i \in [n], \quad u_i \leq \sum_{j \in [m]} v_{ij} x_{ij} + v_i \tag{114}$$

$$\forall i \in [n], \quad \sum_{j \in [m]} x_{ij} \leq 1 \tag{115}$$

$$\forall i \in [n], j \in [m] \quad x_{ij}, v_i \geq 0 \tag{116}$$

Dual

$$\min_{\boldsymbol{p}, \boldsymbol{\beta}} \sum_{j=1}^{m} p_j - \sum_{i=1}^{n} b_i \log(\beta_i) \tag{117}$$

$$\forall i \in [n], j \in [m] \quad p_j \geq v_{ij} \beta_i \tag{118}$$

$$\forall i \in [n], \quad \beta_i \leq 1 \tag{119}$$

The proof is skipped as it is similar to that of theorem 4.6.

4.4 First and Second Welfare Theorems for Fisher Market Walrasian Equilibria

One might ask why Walrasian equilibria are important and that question can be answered by the first and second Welfare theorem of economics. Namely, the Walrasian equilbirium allocations turn out to be efficient and for any pareto-optimal allocation, one can calculate prices to obtain a Walrasian equilibrium. The importance of the first second welfare theorems is that they essentially support Adam Smith's invisible hand hypothesis. That is, there exists an equilibrium, namely the Walrasian one, that is efficient assuming self-interested, i.e., rational, agents. We first recall the definition of Pareto-optimal.

[7] It is ill-defined in the sense that quasilinear utility functions include a price variable, while the convex programming solution to UMP (i.e., the Eisenberg-Gale program) does not; it concerns only utilities, allocations, and budgets.

Definition 4.11 (Pareto-Optimal)**.** *An allocation $\boldsymbol{X}$ is said to be Pareto-optimal if there exists no other feasible allocation that makes some buyer better-off without making any other buyers worse-off, that is, let $\boldsymbol{X}$ be any feasible allocation then an allocation $\boldsymbol{X}^*$ is Pareto-optimal if:*

$$\exists i \in [n], \quad u_i(\boldsymbol{x}_i) > u_i(\boldsymbol{x}_i^*) \implies \exists k \in [n], \quad u_i(\boldsymbol{x}_i^*) > u_i(\boldsymbol{x}_i) \tag{120}$$

The first welfare theorem states that any Walrasian equilibrium allocation is Pareto-optimal.

Theorem 4.12 (First Welfare Theorem - Fisher Market)**.** *If $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ is a Walrasian equilibrium then it is also pareto-optimal.*

Proof. Let $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ be a Walrasian equilibrium. By way of contradiction, assume that there exists another outcome $(\boldsymbol{X}, \boldsymbol{p})$ for which we have $\forall i \in [n], \;\; u_i(\boldsymbol{x}_i) \geq u_i(\boldsymbol{x}_i^*)$ and $\exists i \in [n], \;\; u_i(\boldsymbol{x}_i) > u_i(\boldsymbol{x}_i^*)$. Since utility functions are non-satiated and Walrasian equilibria are utility maximizing, then we must have that $\forall i \in [n], \;\; \boldsymbol{p} \cdot \boldsymbol{x}_i \geq \boldsymbol{p} \cdot \boldsymbol{x}_i^*$ and $\exists i \in [n], \;\; \boldsymbol{p} \cdot \boldsymbol{x}_i > \boldsymbol{p} \cdot \boldsymbol{x}_i^*$. That is in order to achieve a higher utility with another allocation that is not a Walrasian equilibrium, buyers need to be on an indifference curve that intersects with a budget constraint curve further way from the origin. Since at a Walrasian equilibrium buyers already were spending their entire budget, any outcome that pareto-dominates the Walrasian Equilibrium outcome $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ must be infeasible. □

The second welfare theorem states that for an Pareto-optimal allocation, we can compute prices such as to obtain a Walrasian equilibrium.

Theorem 4.13 (Second Welfare Theorem - Fisher Market)**.** *Let $(U, \boldsymbol{b})$ be a Fisher Market, if $\boldsymbol{X}^*$ is pareto-optimal allocation for $(U, \boldsymbol{b})$ then there exists a price vector $\boldsymbol{p}^*$ such that $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ is a Walrasian equilibrium. That is, any Pareto-optimal outcome can be represented as Walrasian outcome.*

Skipping proof as it is relatively involved.

Note that the second welfare theorem does not say that for any Fisher market, every Pareto optimal allocation is a Walrasian equilibrium. Rather, it says that for any Pareto optimal allocation of a Fisher Market there is a way to re-distribute resources through prices that makes the allocation a Walrasian equilibrium outcome.

4.5 Stability of Walrasian Equilibria in Fisher Markets

As we have shown, the first and second welfare theorem show that market equilibria are efficient, however, this is only under the assumption that markets reach such equilibria. The question of whether if there exist real-world-like market dynamics/processes that lead to price adjustments that lead to market equilibria is known as the is **stability question**.

Although programs such the Eisenberg-Gale program allow us to compute Fisher market equilibria, via interior point methods for instance, these computation techniques cannot be seen as market processes as they are centralized and have no economic interpretation. As a result, it is crucial for economists and computer scientists alike to come up with distributed economic processes that reach equilibrium and that immitate real-world phenomena. The question could be argued to date back to Léon Walras, a French economist, who in 1874 conjectured that a natural price-adjustment process he called **tâtonnement**, an algorithm representative of market behavior, would converge to equilibrium prices [17]. An early positive result in this vein was provided by Arrow, Block and Hurwicz, who showed that a continuous version of tâtonnement converges in markets with an aggregate demand function that satisfies the **weak gross substitutes (WGS)** property [3].

Another reason to be interested in the question of stability is an "engineering" one. For instance, the problem of allocating bandwidth on a network can be solved in a decentralized manner using tâtonnement-like protocols. Additionally, with the rise of blockchain decentralized finance technology, one can imagine that eventually centralized market places might get replaced with exchange protocols, e.g. tâtonement, that are guaranteed to be optimal and that eliminate middle-men, i.e., market place managers.

Going back to tâtonnement in Fisher markets, more recently, Cole and Fleischer, and Cheung et al. showed the fast convergence of tâtonnement in Fisher markets where the buyers' utility functions satisfy the weak gross substitutes and the **constant elasticity of substitution (CES)** properties respectively, a subset of the class of CCH utility functions [6, 7, 8]. Although tâtonnement has also been criticized for being a centralized process, which therefore cannot suitably model real-world markets, Cole and Fleischer argued for the plausibility of tâtonnement by proving that it is an abstraction for in-market processes in a real-world-like model called the ongoing market model [7, 8]. Additionally, the plausibility of tâtonnement as a natural price-adjustment process has been supported by Gillen et al., who demonstrated the predictive accuracy of tâtonnement in off-equilibrium trade settings [12]. We summarize the convergence results for tâtonnement in Fisher markets in fig. 1.

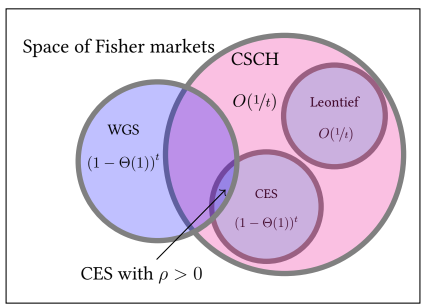

Figure 1: The convergence rates of tâtonnement for different classes of utility functions. We color previous contributions blue, and our contributions red, i.e., we study Fisher markets for buyers with CSCH utilities. We note that the convergence rate for CES and WGS markets does not apply to markets where buyers have linear utilities.

The tâtonnement process can be seen as a continuous or a discrete process. Let $G : \mathbb{R}^m \to \mathbb{R}^m$ be a monotone function. The **excess demand**, $z : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, in a Fisher market $(U, \boldsymbol{b})$ is defined as the difference between the demand for each good and the supply of each good, that is:

$$z(\boldsymbol{p}) = \sum_{i \in [n]} d_i(\boldsymbol{p}, b_i) - \mathbf{1}_m \tag{121}$$

where $\mathbf{1}_m$ is the vector of ones of size m and $\sum_{i\in[n]} d_i(\boldsymbol{p}, b_i) - \mathbf{1}_m = \left\{\boldsymbol{x} - \mathbf{1}_m | \boldsymbol{x} \in \sum_{i\in[n]} d_i(\boldsymbol{p}, b_i)\right\}$.

Definition 4.14 (Continuous Tâtonnement)**.** *The continuous tâtonnement process is given by the dyanmical system with phase space $\mathbb{R}^m_+$ and evolution function:*

$$\dot{\boldsymbol{p}} = G \circ z(\boldsymbol{p}(t)) \tag{122}$$

Definition 4.15 (Discrete Tâtonnement)**.** *The discrete tâtonnement process is given by the dynamical system with phase space $\mathbb{R}^m$ and evolution function:*

$$\boldsymbol{p}(t+1) = \boldsymbol{p}(t) + G \circ z(\boldsymbol{p}(t)) \tag{123}$$

5 The Pari-Mutuel Betting Model

We now digress to introduce an interesting connection between Fisher markets and a betting market model name the Pari-Mutual betting model first rigorously analyzed by Edmund Eisenberg and David Gale in 1959. We note that Pari-mutuel model and the Fisher market were both invented around the 1970s, but the Eisenberg-Gale program was first proposed by Eisenberg and Gale to solve for an equilibrium of the Pari-Mutuel betting model. In 1867, Spanish entrepreneur Joseph Oller invented parimutuel betting, a form of wagering that is still popular today and handles billions of waged dollars every year! The pari-mutuel model is a system of betting in which all bets are placed on a set of possible exclusive outcomes. Once the outcome is determined the bettors who bet on the winning outcome split the total amount wagered in proportion to the size of their wagers. To explain this system better we will consider the application of pari-mutuel betting to betting in horse races.

In this setting, each bettor bets on a horse. The house collects the bets and pays the participants that bet on the winning horse the total amount of money collected multiplied by the proportion of each winner's bet in the total amount bet on the winning horse.

The first principle of the pari-mutuel model is that you should not try to bet on the horse that has the highest chance of winning, but on the horse that has the best pay out ratio relative to your belief of its chance of winning. The reason for this is simply that as more people bet on the same horse the likelihood of any bettor making a profit goes to 0. To better illustrate this principle, consider the trivial case when every agent bets all their money on the same horse. If that horse ends up winning, every bettor gets back the amount of money he/she bet but makes no profit whatsoever. Since, the goal of betting is to make a profit, a bettor is better off guessing a winning horse on which only a small amount of money is bet.

5.1 Model

We now introduce the mathematical model proposed by Edmund Eisenberg and David Gale[8] [11]. The **Pari-Mutuel betting model** consists of:

- n horses labelled, $H_1, \ldots, H_n$
- m bettors labelled $B_1, \ldots, B_m$
- Each bettor $B_i, \forall i \in [m]$, has a budget b_i. We assume that the sum of the budgets is equal to one, i.e., $\sum_{i=1}^{m} b_i = 1$
- Matrix of *subjective opinions* of winning, where each entry $(i, j)^{th}$ is the prior that each bettor B_i has on the probability of winning of horse H_j such that $\boldsymbol{P} = [p_{ij}]_{i\in[m],j\in[n]}$
- WLOG, we assume that each horse i has at least one subjective opinion of winning that is strictly positive, i.e. $p_{ij} > 0$. Otherwise, we could simply not consider that horse in the betting process because no one will bet money on it (since no bettor believes that the horse will win).

5.2 Outcome of the Model

Each bettor B_i bets part of their budget on any horse H_j. Let $\boldsymbol{\beta} = [\beta_{ij}]_{i\in[m],j\in[n]}$ be the **bet allocation** matrix s.t. $\beta_{ij} \geq 0$ denotes the amount that bettor B_i bet on horse H_j. After the bets have been made, it is possible to

[8] Fun fact: Gale was a Professor in the applied math department here at Brown!

determine an aggregation of all bettors' opinions to determine the aggregated winning probability of each horse. Let $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)^T$ be the **final track probabilities** vector which represents the probability of each horse winning based on the bets have been made.[9] A pari-mutuel outcome is a tuple $(\boldsymbol{\beta}, \boldsymbol{\pi})$ consisting of bet allocations and final track probabilities.

Any pari-mutuel outcome must satisfy the **budget relation**: $\sum_{j=1}^{m} \beta_{ij} \leq b_i$ (i.e., the sum of the bets of each bettor is equal to their budget) and the **pari-mutuel condition**: $\sum_i \beta_{ij} = \pi_j$ (i.e., the sum of the bets on each horse is equal to their final track probability). The pari-mutuel condition simply arithmetizes the main principle behind the model which is that the probability of a horse winning is proportional to the amount bet on that horse. Note that, since we assumed that the sum of all of the budgets is equal to 1, in this case the proportonality constant is exactly 1.

As stated before, the goal of each bettor is to bet on the horse that has the best pay out ratio relative to their belief of the horse's winning odds. Since the bettors make a bigger profit when the winning horse has less money bet on it, the goal of each bettor can be modelled as betting on the horses with the highest ratio of subjective probability of winning to final track probability[10]. A pari-mutuel outcome $(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*)$ in which the bettors bet in this manner is said to be **expectation maximizing**. We arithmetize the expectation maximization condition as follows:

$$\text{if } \mu_i = \max_s \frac{p_{is}}{\pi_s^*} \text{ and } \beta_{ij}^* > 0, \text{ then } \mu_i = \frac{p_{ij}}{\pi_j^*} \tag{124}$$

A pari-mutuel outcome is **optimal** iff bettors are expectation maximizing and the outcome respects the budget relation as well as the pari-mutuel condition.

Now, observe that the final track probabilities cannot be determined before the bettors have put their money on their horses. However, bettors can also not maximize their expectation without knowing the final track probabilities! So one question to ask is whether if an optimal outcome of the pari-mutuel model exists at all!

5.3 Optimal Solution

We now introduce the Eisenberg-Gale convex program.

$$\max_{\boldsymbol{\xi}} \sum_{i=1}^{m} b_i \log\left(\sum_{j=1}^{n} p_{ij}\xi_{ij}\right) \tag{125}$$

$$\xi_{ij} \geq 0 \tag{126}$$

$$\sum_{i=1}^{m} \xi_{ij} = 1 \tag{127}$$

The optimal output $\bar{\boldsymbol{\xi}} = \left[\bar{\xi}_{ij}\right]_{i\in[m],j\in[n]}$ to this convex program can be used to calculate the optimal allocation $(\boldsymbol{\beta}, \boldsymbol{\pi})$. This is also a proof of the existence an optimal allocation for the pari-mutuel problem (given that our claim in the previous sentence is correct) since any convex program is guaranteed to have a minimum.

Theorem 5.1. *Let $\bar{\boldsymbol{\xi}}$ be a solution of the Eisenberg-Gale program. An optimal allocation $(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*)$ for the pari-*

[9]This vector of probability represents the aggregation of all bettors subjective probabilities into a unique probability vector.

[10]Remember that a lower final track probability means less money bet on a horse due to the pari-mutuel condition

mutuel market can be calculated as:

$$\pi_j^* = \max_i \frac{b_i p_{ij}}{\sum_{s=1}^{n} p_{is}\bar{\xi}_{is}} \tag{128}$$

$$\beta_{ij}^* = \bar{\xi}_{ij}\pi_j^* \tag{129}$$

Proof. First, we claim the following, which we use in the rest of the proof:

$$\text{if } \bar{\xi}_{ij} > 0 \text{ then } \pi_j = \frac{b_i p_{ij}}{\sum_{s=1}^{n} p_{is}\bar{\xi}_{is}} \tag{130}$$

To see this, suppose that this claim is false and that for some i, j, $\bar{\xi}_{ij} > 0$ and that $\pi_j > \frac{b_i p_{ij}}{\sum_{s=1}^{n} p_{is}\bar{\xi}_{is}}$. By definition of π_j, we then have a $\bar{\xi}_{kj}$ for some k which gives $\pi_j = \frac{b_k p_{kj}}{\sum_{s=1}^{n} p_{ks}\bar{\xi}_{ks}} > \frac{b_i p_{ij}}{\sum_{s=1}^{n} p_{is}\bar{\xi}_{is}}$. This then implies that we could decrease $\bar{\xi}_{ij}$ and increase $\bar{\xi}_{kj}$ to increase the objective function of the Eisenberg-Gale program. This, however, is a contradiction since $\bar{\xi}$ maximizes the Eisenberg-Gale program. In other words, this fact simply comes from the fact that $\bar{\xi}$ maximizes the Eisenberg-Gale program.

Budget constraint condition: Combining conditions (129) and (130), we get:

$$\beta_{ij}^* = \bar{\xi}_{ij}\pi_j^* \tag{131}$$

$$= \bar{\xi}_{ij}\frac{b_i p_{ij}}{\sum_{s=1}^{n} p_{is}\bar{\xi}_{is}} \tag{132}$$

$$= b_i \frac{p_{ij}\bar{\xi}_{ij}}{\sum_{s=1}^{n} p_{is}\bar{\xi}_{is}} \tag{133}$$

Summing the above on j, we get: $\sum_{j=1}^{n} \beta_{ij}^* = b_i \frac{\sum_{j=1}^{n} p_{ij}\bar{\xi}_{ij}}{\sum_{s=1}^{n} p_{is}\bar{\xi}_{is}} = b_i$. This confirms the budget constraint condition.

Pari-mutuel Condition: By constraint (127), we know that $\sum_{i=1}^{m} \bar{\xi}_{ij} = 1$. Hence, summing up β_{ij}^* on i we recover the final track probabilities, i.e., $\sum_{i=1}^{m} \beta_{ij}^* = \sum_{i=1}^{m} \bar{\xi}_{ij}\pi_j^* = \pi_j^*$. This confirms that the allocation respects the pari-mutuel condition.

Expectation Maximization

From (128), we know that:

$$\pi_j^* = \max_i \frac{b_i p_{ij}}{\sum_{s=1}^{n} p_{is}\bar{\xi}_{is}} \tag{134}$$

$$\frac{1}{\pi_j} = \min_i \frac{\sum_{s=1}^{n} p_{is}\bar{\xi}_{is}}{b_i p_{ij}} \tag{135}$$

$$\frac{p_{ij}}{\pi_j} = \min_i \frac{\sum_{s=1}^{n} p_{is}\bar{\xi}_{is}}{b_i} \tag{136}$$

$$\frac{p_{ij}}{\pi_j} \le \frac{\sum_{s=1}^{n} p_{is}\bar{\xi}_{is}}{b_i} \tag{137}$$

Recall from (124) that:

$$\mu_i = \max_s \frac{p_{is}}{\pi_s^*} \tag{138}$$

Substituting π_s^* with the expression from (128), we get:

$$\mu_i = \max_s p_{is} \frac{\sum_{k=1}^{n} p_{ik}\bar{\xi}_{ik}}{b_i p_{is}} \tag{139}$$

$$= \max_s \frac{\sum_{k=1}^{n} p_{ik}\bar{\xi}_{ik}}{b_i} \tag{140}$$

$$= \frac{\sum_{k=1}^{n} p_{ik}\bar{\xi}_{ik}}{b_i} \tag{141}$$

Since we assumed that there exists at least one entry in each column of $\boldsymbol{P}$ that is strictly positive, we know that each π_j is positive[11] Then, combining facts (129), (140), (137) and (130), we get exactly (124).

□

5.4 Connecting the Pari-Mutuel Model and the Fisher Market

We will now assume that the utility functions of the buyers are linear to provide an interesting connection between the optimal outcome of the Pari-Mutuel model and the Fisher Market. Let each buyer have preferences over goods represented as a vector of values $\boldsymbol{v}_i \in \mathbb{R}^m$, Linear utilities are defined as:

$$\forall i \in [n], u_i(\boldsymbol{x}_i) = \sum_{jin[m]} v_{ij} x_{ij} \tag{142}$$

It turns out that any equilibrium allocation of the fisher market is captured by the solution to the Eisenberg-Gale program. The corespondance between the variables in the both models is given in the table below:

Pari-Mutuel	**Fisher**
b_i (bettor budget)	B_i (buyer budget)
ξ_{ij} of B_i's bet in π_j)	X_{ij} (allocation of good j to buyer i)
p_{ij} (subjective probability)	v_{ij} (valuation)
π_j (final track probability)	p_j (price)
β_{ij} (bet of B_i on H_j)	$X_{ij}p_j$ (s pending of buyer i on good j)

This equivalence allows us to use the Eisenberg-Gale program to solve linear Fisher markets. It turns out that one can actually generalize the Eisenberg-Gale program to all homothetic Fisher markets.

[11] Observe the objective function of the Eisenberg-Gale program to convince yourself about this fact.

6 Arrow-Debreu Model of a Competitive Economy

The general equilibrium model of a competitive economy, also known as the Arrow-Debreu model, establishes the existence of a general equilibrium, that is prices, consumption and production that maximize firms profits, consumers' utilities and clears the market [2].

6.1 Model

An **Arrow-Debreu model (of a competitive economy)** consists of:

1. Finite set of l commodities (this can include raw goods, intermediate goods and labor)
2. Finite set of n production units (i.e., firms). Each firm $j \in [n]$ has:
   - a set of possible productions Y_j. An element $\boldsymbol{y}_j \in Y_j$ is a vector in $\mathbb{R}^l$. Positive elements of this vector are outputs while negative elements are inputs.
3. Finite set of m consumption units (i.e., agents/consumers). Every agent $i \in [m]$ has:
   - a set of possible consumptions X_i. An element $\boldsymbol{x}_i \in X_j$ is a vector in $\mathbb{R}^l$. Positive elements of this vector are commodities consumed while negative elements are the labor service that a consumer provides. We asumme that the labor that a consumer can provide is upperbounded.
   - a contractual claim to a share of the profits of each firm $\boldsymbol{\alpha}_i = (\alpha_{i1}, \ldots, \alpha_{in})$
   - an endowment of commodities $\boldsymbol{e}_i = (e_{i1}, \ldots, e_{il})$
   - a utility function $u_i : \mathbb{R}^l_+ \to \mathbb{R}$ that gives that the utility that an agent derives from a bundle of commodities.

6.2 Model Outcome

The price space is $P = \left\{\boldsymbol{p} \mid \ p_h \geq 0, \ \sum_{h=1}^{l} p_h = 1\right\}$. Commodities are assigned **prices** $\boldsymbol{p} \in P$. An outcome of the model is a tuple $(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m, \boldsymbol{p}) \in Y_1 \times \cdots \times Y_n \times X_1 \times \cdots \times X_m \times P$.

An outcome $(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m, \boldsymbol{p}) \in Y_1 \times \cdots \times Y_n \times X_1 \times \cdots \times X_m \times P$ is feasible iff the value of the consumption of agents is less than or equal to their income, i.e., $\forall i \in [m], \ \ \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p} + \sum_{j=1}^{n} \alpha_{ij} \boldsymbol{y}_j \cdot \boldsymbol{p}$.

6.3 Equilibrium

An outcome $(\boldsymbol{y}_1^*, \ldots, \boldsymbol{y}_n^*, \boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_m^*, \boldsymbol{p}^*) \in Y_1 \times \cdots \times Y_n \times X_1 \times \cdots \times X_m \times P$ is an equilibrium iff:

1. Firms maximizes profit:
   $\forall j \in [n], \quad \boldsymbol{y}_j^*$ maximizes $\boldsymbol{p}^* \cdot \boldsymbol{y}_j$ over Y_j
2. Consumers maximize utility:
   $\forall i \in [m] \quad \boldsymbol{x}_i^*$ maximizes $u_i(\boldsymbol{x}_i; \boldsymbol{v}_i)$ over the set $\left\{\boldsymbol{x}_i \mid \ \boldsymbol{x}_i \in X_i \ \ \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^* + \sum_{j=1}^{n} \alpha_{ij} \boldsymbol{y}_j^* \cdot \boldsymbol{p}^*\right\}$
3. The markets clear and goods that are not demanded are priced at 0:
   $\sum_{i=1}^{m} \boldsymbol{x}_i^* - \sum_{j=1}^{[n]} \boldsymbol{y}_j^* - \sum_{i=1}^{m} \boldsymbol{e}_i \leq 0$, and $\boldsymbol{p}^* \cdot \left(\sum_{i=1}^{m} \boldsymbol{x}_i^* - \sum_{j=1}^{[n]} \boldsymbol{y}_j^* - \sum_{i=1}^{m} \boldsymbol{e}_i\right) = 0$

Note that for the last condition, we need both mathematical statements since the first condition coupled with the second one ensures that if a good is under-demanded that it is priced at 0. (if it confuses you, the second part is more of a technical statement in a way because we assume that the prices cannot be negative) We now present the Arrow-Debreu Theorem due to Nobel prize laureate economists Kenneth J. Arrow and Gerard Debreu.

Theorem 6.1 (The Arrow-Debreu Theorem I). *Suppose that the following conditions are satisfied:*

1. X_i *is closed and convex for all* $i \in [m]$
2. Y_j *is closed and convex for all* $j \in [n]$
3. *All agents have a consumption that is stricly less than their endowment, i.e., for all agents* $i \in [m]$, $\exists \boldsymbol{x}_i \in X_i, \quad \boldsymbol{x}_i < \boldsymbol{e}_i$
4. u_i *is continuous*
5. u_i *is quasi-concave, i.e.,* $\forall \lambda \in (0,1), \ \ u_i(\lambda x + (1-\lambda) y) \geq \min \{u_i(x), u_i(y)\}$,
6. u_i *is non-satiated, i.e.,* $\forall \boldsymbol{x} \in \mathbb{R}^l, \ \ \epsilon > 0, \ \ \exists \boldsymbol{y} \in \left\{\boldsymbol{y} | \boldsymbol{y} \in \mathbb{R}^l, ||\boldsymbol{y} - \boldsymbol{x}|| \leq \epsilon \right\}, \quad u_i(\boldsymbol{y}) > u_i(\boldsymbol{x})$

Then an equilibrium outcome of the Arrow-Debreu Model exists

Note that when $X = \mathbb{R}^l_+$ (which is both an open and closed), then assumption 3 reduces to $\forall i \in [m], h \in [l], e_{ih} > 0$.

Proof Sketch. We provide a sketch of the proof as it is a very involved proof.

Define the following optimization program whose output is the utility maximizing and budget constrained consumption bundle of consumer i (i.e., the demand of consumer i):

$$D^i(\boldsymbol{p}) = \left\{ \begin{array}{c} \arg\max_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i) \\ \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p} + \sum_{j=1}^{n} \alpha_{ij} \boldsymbol{y}_j \cdot \boldsymbol{p} \\ \boldsymbol{x}_i \in X_i \end{array} \right\} \tag{143}$$

where $\boldsymbol{y}_j$ is chosen arbitrarily.

By the assumptions of the theorem u_i is continuous, quasi-concave, and non-satiated and X_i is compact (because X_i is a closed subspace of $\mathbb{R}^l$), this means that the output of this program is unique for any input price vector $\boldsymbol{p}$ (meaning that it can be considered as regular function). Furthermore, these assumptions allow us to use a theorem called the maximum theorem that tells us that this function is continuous in its arguments (i.e., prices).

Define the following optimization program whose output is the profit maximizing production of firm j:

$$S^j(\boldsymbol{p}) = \left\{ \begin{array}{c} \arg\max_{\boldsymbol{y}_j} \boldsymbol{y}_j \cdot \boldsymbol{p} \\ \boldsymbol{y}_j \in Y_j \end{array} \right\} \tag{144}$$

By the assumptions of the theorem Y_j is compact (because $Y_{[n]}$ is a closed subspace of $\mathbb{R}^l$), since Y_j is convex, then the output of this program is unique (meaning that it can be treated like a regular function). Furthermore, by the maximum theorem, this means that this function is continuous in its arguments (i.e., prices).

Define the excess demand function for the economy that returns a vector of the differences in the supply and demand of each commodity:

$$Z(p) = \sum_{i\in[m]} D^i(\boldsymbol{p}) - \sum_{j\in[n]} S^j(\boldsymbol{p}) - \sum_{i\in[m]} \boldsymbol{e}_i \tag{145}$$

$$Z : P \to \mathbf{R}^N \tag{146}$$

Now define the function $T : P \to P$, that mimics a fictional auctioneer trying to bring the economy into an equilibrium by adjusting the prices based on the excess demand. More specifically this function is calculated as:

$$T_h(\boldsymbol{p}) = \frac{\max\left[0, p_h + \gamma_k Z_k(\boldsymbol{p})\right]}{\sum_{k\in[l]} \max\left[0, p_k + \gamma_k Z_k(\boldsymbol{p})\right]} \tag{147}$$

where is an arbitrary constant $\gamma_k > 0$.

The function T is a price adjustment function. It raises the relative price of goods in excess demand and reduces the price of goods in excess supply while keeping the price vector on the simplex. This function is also continuous since the demand and supply functions are continuous which implies that the excess demand function is continuous and since the $\max$ operator preserves continuity, T must also be continuous. Since T is continuous and maps from a convex compact set back to itself, by Brouwer's fixed point theorem, it has a fixed point. By the definition of the function this fixed point must be an equilibrium.[12]

□

Assumption 3 that every consumer has an endowment such that they could consume it and have left over endowment to sell is a very strong one. As a result Arrow and Debreu provide a second theorem with additional assumptions that gets rid of that assumption.

Theorem 6.2. ***The Arrow-Debreu Theorem II***

Suppose that the following conditions are satisfied:

X_i is closed and convex for all $i \in [m]$

2. *Y_j is closed and convex for all $j \in [n]$*
3. *Each agent has at least one good that they are endowed with, for which they have a consumption that does not consume that good entirely, i.e., $\forall i \in [m], \exists h \in [l], \quad x_{ih} < e_{ih}$.*
4. *There exists a consumption for all agents such the supply of goods is strictly greater than demand. More formally, Let $X = \left\{\boldsymbol{X} | \boldsymbol{X} = \sum_{i\in i} \boldsymbol{x}_i,\ \text{where } \boldsymbol{x}_i \in X_i\right\}$ and $Y = \left\{\boldsymbol{y} | \boldsymbol{y} = \sum_{j\in[n]} \boldsymbol{y}_j,\ \text{where } \boldsymbol{y}_j \in Y_j\right\}$ $\exists \boldsymbol{X} \in X, \boldsymbol{y} \in Y$, then $\exists \boldsymbol{X} \in X, \boldsymbol{y} \in Y, \quad \boldsymbol{X} < \boldsymbol{y} + \sum_{i\in[m]} \boldsymbol{e}_i$*
5. *u_i is continuous*

[12] Some of the mathematical details were skipped for brevity, you can find more detail notes here

6. u_i *is quasi-concave, i.e.,* $\forall \lambda \in (0,1), \;\; u_i(\lambda x + (1-\lambda)y) \geq \min\{u_i(x), u_i(y)\}$,

7. u_i *is non-satiated, i.e.,* $\forall \boldsymbol{x} \in \mathbb{R}^l, \;\; \epsilon > 0, \;\; \exists \boldsymbol{y} \in \left\{\boldsymbol{y} | \boldsymbol{y} \in \mathbb{R}^l, ||\boldsymbol{y} - \boldsymbol{x}|| \leq \epsilon\right\}, \quad u_i(\boldsymbol{y}) > u_i(\boldsymbol{x})$

Then an equilibrium outcome of the Arrow-Debreu Model exists

7 Arrow-Debreu Exchange Economy Model

7.1 Model

An **Arrow-Debreu Exchange economy model** consists of:

1. Finite set of m goods.
2. Finite set of n traders (i.e., agents/consumers). Every trader $i \in [n]$ has:
   - a set of possible consumptions $X_i \subseteq \mathbb{R}^m_+$. We denote $X = \bigtimes_{i \in [n]} X_i$
   - an endowment of goods $\boldsymbol{e}_i = (e_{i1}, \ldots, e_{im}) \in \mathbb{R}^m$. We denote $\boldsymbol{E} = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)^T$
   - a utility function $u_i : \mathbb{R}^m \to \mathbb{R}$.

An instance of an Arrow-Debreu market is given by a tuple $(n, m, X, U, \boldsymbol{E})$. When clear from context, we abbreviate $(U, \boldsymbol{E})$.

7.2 Walrassian/Competitive Equilibrium

An **allocation** $\boldsymbol{X} \in X$ is a map from goods to buyers, represented as a matrix, s.t. $x_{ij} \geq 0$ denotes the amount of good $j \in [m]$ allocated to buyer $i \in [n]$. The price space is $P = \left\{\boldsymbol{p} \mid \; p_j \geq 0, \; \sum_{j=1}^m p_j = 1\right\}$. Goods are assigned **prices** $\boldsymbol{p} \in P$. Goods are assigned **prices** $\boldsymbol{p} \in P$. An outcome of the model is a tuple $(\boldsymbol{X}, \boldsymbol{p}) \in X \times P$.

An outcome $(\boldsymbol{X}^*, \boldsymbol{p}^*) \in X \times P$ is an equilibrium iff:

1. Traders maximize utility:
   $\forall i \in [n] \;\;\; \boldsymbol{x}_i^*$ maximizes $u_i(\boldsymbol{x}_i)$ over the set $\{\boldsymbol{x}_i \mid \; \boldsymbol{x}_i \in X_i \;\; \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^*\}$
2. The markets clear and goods that are not demanded are priced at 0:
   $\sum_{i=1}^m \boldsymbol{x}_i^* \leq \sum_{i=1}^m \boldsymbol{e}_i$ and $\boldsymbol{p}^* \cdot (\sum_{i=1}^m \boldsymbol{x}_i^* - \sum_{i=1}^m \boldsymbol{e}_i) = 0$

Theorem 7.1 (Arrow-Debreu Exchange Theorem)**.** *Suppose that the following conditions are satisfied:*

1. X_i *is closed and convex for all* $i \in [n]$
2. *Each agent has a consumption* $\boldsymbol{x}_i \in X_i$ *that is strictly less than their endowment for all goods, i.e.,* $e_{ij} > x_{ij}$ *for all* $i \in [n]$ *and* $j \in [m]$.
3. u_i *is continuous*
4. u_i *is quasi-concave, i.e.,* $\forall \lambda \in (0,1), \;\; u_i(\lambda x + (1-\lambda)y) \geq \min\{u_i(x), u_i(y)\}$
5. u_i *is locally non-satiated, i.e.,* $\forall \boldsymbol{x} \in X_i, \;\; \forall \epsilon > 0, \;\; \exists \boldsymbol{y} \in B_\epsilon(\boldsymbol{x}), u_i(\boldsymbol{y}) > u_i(\boldsymbol{x})$

Then an equilibrium outcome of the Arrow-Debreu Model exists.

Note that when $X = \mathbb{R}^l_+$ (which is both an open and closed), then assumption 3 reduces to $\forall i \in [m], h \in [l], e_{ih} > 0$. The proof of the above theorem is a direct corollary as the Arrow-Debreu theorem for competitive economies since the Arrow-Debreu Exchange model is a special case (actually a later paper proved that both models are equivalent!) of the Arrow-Debreu Competitive Economy model and the result from the Arrow-Debreu theorem directly applies to the exchange model.

Theorem 7.2 (The Arrow-Debreu Exchange Theorem II). *Suppose that the following conditions are satisfied:*

1. X_i *is closed and convex for all* $i \in [n]$
2. *Each consumer is endowed with at least one good for which they have a consumption that will not consume their entire endowment of that good, i.e.,* $\forall i \in [n], \exists j \in [m], \boldsymbol{x}_i \in X_i$, *s.t.,* $x_{ij} < e_{ij}$.
3. *There exists a consumption for all consumers such that they can consume less than the total endowment, i.e.* $\exists \boldsymbol{X} \in X$, *s.t.* $\sum_{i\in[n]} \boldsymbol{x}_i < \sum_{i\in[n]} \boldsymbol{e}_i$.
4. u_i *is continuous*
5. u_i *is quasi-concave, i.e.,* $\forall \lambda \in (0,1),\ \ u_i(\lambda x + (1-\lambda)y) \geq \min\{u_i(x), u_i(y)\}$
6. u_i *is locally non-satiated, i.e.,* $\forall \boldsymbol{x} \in X_i,\ \ \forall \epsilon > 0,\ \ \exists \boldsymbol{y} \in B_\epsilon(\boldsymbol{x}), u_i(\boldsymbol{y}) > u_i(\boldsymbol{x})$

Then an equilibrium outcome of the Arrow-Debreu Model exists.

7.3 Connecting Fisher Markets and Arrow-Debreu Exchange Economies

Theorem 7.3. *An instance* $(U, \boldsymbol{b})$ *of a Fisher market can be cast as an instance* $(U', \boldsymbol{E})$ *of an Arrow-Debreu Exchange market.*

Proof. We provide a reduction from an arbitrary instance of the Fisher Market to an instance of the Arrow-Debreu Exchange Market. Let $(U, \boldsymbol{b})$ be an instance of a Fisher Market.

We build the following Arrow-Debreu exchange market with $n+1$ consumers and $m+1$ goods. More specifically, we add an $(m+1)^{th}$ commodity which is money, and a new $(n+1)^{th}$ artificial consumer which initially will have all m goods, and is interested only in money.

We set the initial endowments $\boldsymbol{E}$ of the consumers in this construction of the Arrow-Debreu Exchange model as follows:

Consumer	Commodity 1	...	Commodity m	Commodity $m+1$
1	0	...	0	b_1
2	0	...	0	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	0	...	0	b_n
$n+1$	1	...	1	0

The consumption set of the consumers are set as follows:

$$\forall i \in [n+1], \qquad X_i = \left\{ \boldsymbol{x}_i \mid \forall h \in [m+1],\ \ 0 \leq x_{ih} \leq \sum_{i \in [n+1]} e_{ih} \right\} \tag{148}$$

The new utility functions $u^{'}$ of the consumers are set as follows:

- The first n consumers, derive the utility given by U from the first m commodities and a utility of 0 for the $(m+1)^{th}$ commodity (i.e., money), i.e., $\forall i \in [n], u_i^{'}(\boldsymbol{x}_i) = u_i(\boldsymbol{x}_{i1}, \dots, \boldsymbol{x}_{im})$
- The $(n+1)^{th}$ consumer, derives no utility from the first m commodities and a utility of x_{n+1m+1} from the $(m+1)^{th}$ commodity (i.e., money) where x_{n+1m+1} is the amount of money that the artificial consumer is allocated, i.e., $u_{n+1}^{'}(\boldsymbol{x}_{n+1}) = \boldsymbol{x}_{i,m+1}$

We need to now show that the equilibrium prices for the first m goods divided by the equilibrium price of good $m+1$ in this Arrow-Debreu Exchange market correspond exactly to the equilibrium prices for the Fisher market and the allocations of the first m goods correspond exactly to the equilibrium allocations for the Fisher Market.

First, we will show that for the Arrow-Debreu market that we built satisfies conditions of Arrow-Debreu's second theorem so that an equilibrium price vector exists.

The consumption set of the consumers contains all of its end points as it bounded below by 0 and above by the total endowment of agents and those end points are included in the set. As a result it is a closed set. Furtheremore, the set is convex since any convex combination of two arbitrary points in the set belongs to the set. This can be derived by picking two arbitrary points and noticing that any convex combination of those two points will always respect the inequality condition defining the set. The consumption set of all consumers includes the vector of zeros, i.e., consumers consuming nothing. Since every consumer is endowed with the $(m+1)^{th}$ good (i.e., money), condition 2 is also fulfilled. Furthermore, this implies that there exists a consumption vector for consumers as a whole that is the vector of zeros. Since the supply of goods is strictly positive in the entire economy, then condition 3 is also fulfilled. Assuming that the utility functions in the Fisher market were continuous, then the utility functions we build are also continuous. This confirms condition 4. Assuming that the utility functions in the Fisher market were quasi-concave, then the utility functions we build are also quasi-concave since our transformations of the utility functions are monotonous transformations. This confirms condition 5. Assuming that the utility function in the Fisher market were non-satiated then they necessarily also are non-satiated in the Arrow-Debreu market for the first n consumers. Furthermore, the way we built the utility function of consumer $n+1$, getting more of the $(m+1)^{th}$ good strictly increases his utility (i.e., his utility function is monotonic) which implies non-satiation. This confirms condition 6.

Then we show that the equilibrium of the Arrow-Debreu market that we built can be used to compute the equilibrium of the Fisher market. To do so, we show that the equilibrium prices of the Arrow-Debreu market $(U', \boldsymbol{b})$ we built respect the utility maximization (constrained by budget) and market clearance conditions of the Fisher market Walrasian equilibrum.

Budget constraint: Any equilibrium outcome $(\boldsymbol{x}_1^*, \dots, \boldsymbol{x}_m^*, \boldsymbol{p}^*)$ of the Arrow-Debreu model satisfies the feasibility condition, i.e.:

$$\forall i \in [n+1], \qquad \boldsymbol{x}_i^* \cdot \boldsymbol{p}^* \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^* \tag{149}$$

For agents, $1, \dots, n$, since they are only endowed with the $(m+1)^{th}$ good, this condition can be restated as:

$$\forall i \in [n], \qquad \boldsymbol{x}_i^* \cdot \boldsymbol{p}^* \leq b_i p_{m+1}^* \tag{150}$$

Now, if we divide both sides by p_{m+1}^*, we obtain:

$$\forall i \in [n], \qquad \frac{1}{p_{m+1}^*} \boldsymbol{x}_i^* \cdot \boldsymbol{p}^* \leq b_i \tag{151}$$

Let $\boldsymbol{p}^f$ be the equilibrium prices calculated using the equilibrium prices $\boldsymbol{p}^*$ of the Arrow-Debreu exchange model we built, that is, for all $j = 1, \dots, m$, $p_j^f = \frac{p_j^*}{p_{m+1}^*}$. Then, the previous expression becomes:

$$\forall i \in [n], \qquad \boldsymbol{x}_i^* \cdot \boldsymbol{p}^f \leq b_i \tag{152}$$

This confirms that the way we set prices satisfies the budget constraint of consumers in the Fisher market.

Utility Maximization: Firstly, any Arrow-Debreu exchange equilibrium maximizes the utility of the traders. This means that the allocation of goods must also maximize the utility of the buyers in the Fisher market. This is because traders' utility function in the Arrow-Debreu market does not derive any utility for the $(m+1)^{th}$ item (i.e., money). As a result, we know that the allocation of goods in the Arrow-Debreu market maximizes utility based on the m goods. Since for the first m goods, the utility functions of the traders is the same in both the Fisher market and the Arrow-Debreu market, this implies that the first m elements in the equilibrium consumption of the Arrow-Debreu market ensure utility maximization in the Fisher Market too.

Market Clearance Any Arrow-Debreu Equilibrium, is market clearing. to show that the market clearance condition caries to the Fisher market by setting the equilibrium prices in the fisher market as $\forall i \in [n] \boldsymbol{p}^f = \frac{p_j^*}{p_{m+1}^*}$, we will show that the demand set of the buyers is the same for both the Fisher Market and the Arrow-Debreu Market. Let $D^i(\boldsymbol{p})$ be the demand set of buyer i in the fisher market, and let $\Delta^i(\boldsymbol{p})$ be the demand set of the buyer in the Arrow-Debreu market:

$$\forall i \in [n] \qquad \Delta^i(\boldsymbol{p}^*) = \underset{\boldsymbol{x}_i : \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^*}{\arg\max} \; u_i^{'}(\boldsymbol{x}_i) \tag{153}$$

$$\forall i \in [n] \qquad \Delta^i(\boldsymbol{p}^*) = \underset{\boldsymbol{x}_i : \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq b_i p^*_{m+1}}{\arg\max} \; u_i^{'}(\boldsymbol{x}_i) \tag{154}$$

$$\forall i \in [n] \qquad \Delta^i(\boldsymbol{p}^*) = \underset{\boldsymbol{x}_i : \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq b_i p^*_{m+1}}{\arg\max} \; u_i(\boldsymbol{x}_i) \tag{155}$$

$$\forall i \in [n] \qquad \Delta^i(\boldsymbol{p}^*) = \underset{\boldsymbol{x}_i : \frac{1}{p^*_{m+1}} \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq \frac{b_i}{p^*_{m+1}} p^*_{m+1}}{\arg\max} \; u_i(\boldsymbol{x}_i) \tag{156}$$

$$\forall i \in [n] \qquad \Delta^i(\boldsymbol{p}^*) = \underset{\boldsymbol{x}_i : \boldsymbol{x}_i \cdot \boldsymbol{p}^f \leq b_i}{\arg\max} \; u_i(\boldsymbol{x}_i) \tag{157}$$

$$\forall i \in [n] \qquad \Delta^i(\boldsymbol{p}^*) = D^i(\boldsymbol{p}^f) \tag{158}$$

$$\sum_{i \in [n]} \Delta^i(\boldsymbol{p}^*) = \sum_{i \in [n]} D^i(\boldsymbol{p}^f) \tag{159}$$

Hence, since the demand $\sum_{i \in [n]} \Delta^i(\boldsymbol{p}^*)$ ensured that the first m goods cleared in the Arrow-Debreu market (because buyer $n+1$ does not demand any of the first m goods), then $\boldsymbol{p}^f$ must also clear the fisher market.

This means that $\boldsymbol{p}^f$ is the vector of prices that satisfies all competitive equilibrium conditions. Hence, we have shown that we can convert any instance of a fisher market to an arrow-debreu exchange market whose equilibrium maps back to the equilibrium of the fisher market.

□

7.4 Computing Arrow-Debreu Exchange Equilibria

We will now discuss the computational aspects of the Arrow-Debreu model. In a first time, we will discuss solving for Arrow-Debreu equilibria for the case when utility functions of the consumers are Cobb-Douglas, we will then introduce a “natural" process that is guaranteed to converge to equilibrium prices a for a large class of utility functions. A natural process is simply a simple price updating algorithm that can work in a distributed manner or way that simulates real world market behavior. Note that the difficulty of finding Arrow-Debreu equilibria is entirely dependent of the structures of the utility functions of consumers (we will discuss the computational complexity of the Arrow-Debreu model for different classes of utility functions in the next few sections).

7.4.1 Arrow-Debreu - Cobb-Douglas (Eaves 1985)

We now consider the case of the Arrow-Debreu exchange market for which the utility function of the consumers is Cobb-Douglas. For this specific case, Curtis Eaves provided a fast and interesting algorithm [9]. Cobb-Douglas utilities are defined as:

$$\forall i \in [m] \qquad u_i(\boldsymbol{x}_i) = \prod_{h \in [l]} x_{ih}^{v_{ih}} \tag{160}$$

where we assume that $\forall i \in [n], \;\; \sum_{h \in [l]} v_{ih} = 1$

Given prices $\boldsymbol{p}$, we can calculate the demand d^i_h of consumer i for commodity h as follows (the proof follows directly from the closed form formula Marshallian demand for Cobb-Douglas utilities):

$$d_h^i(\boldsymbol{p}) = \frac{v_{ih}(\boldsymbol{p}^T\boldsymbol{e}_i)}{p_h} \tag{161}$$

Using this closed formula, we can calculate the excess demand z_h for commodity h at given prices $\boldsymbol{p}$ as follows:

$$z_h = \sum_{i\in[m]} d_h^i(\boldsymbol{p}) - \sum_{i\in[m]} e_{ih} \tag{162}$$

$$= \sum_{i\in[m]} \frac{v_{ih}(\boldsymbol{p}^T\boldsymbol{e}_i)}{p_h} - \sum_{i\in[m]} e_{ih} \tag{163}$$

Denoting the valuation matrix for all agents by $\boldsymbol{V}$ and the endowment matrix for all agent by $\boldsymbol{E}$, we write the excess demand function z in vector notation to obtain:

$$z(\boldsymbol{p}) = D(\boldsymbol{p})^{-1}\boldsymbol{V}^T\boldsymbol{E}\boldsymbol{p} - \boldsymbol{E}\boldsymbol{j}_m \tag{164}$$

where $D(\boldsymbol{p})$ denotes the matrix whose diagonal entries are the prices for commodities. Note that since $\boldsymbol{V}\boldsymbol{j}_l = \boldsymbol{j}_m$, we can re-write the right hand-side of this expression as follows:

$$z(\boldsymbol{p}) = D(\boldsymbol{p})^{-1}\boldsymbol{V}^T\boldsymbol{E}\boldsymbol{p} - \boldsymbol{E}\boldsymbol{V}\boldsymbol{j}_l \tag{165}$$

Remember that the marshallian demand function calculates the utility maximizing demand of the consumer, hence in order to find equilibrium prices we just need to set the excess demand to obtain maket clearance and get prices for an Arrow-Debreu equilibrium. That is, we are looking for price that satisfy the following condition:

$$D(\boldsymbol{p})^{-1}\boldsymbol{V}^T\boldsymbol{E}\boldsymbol{p} - \boldsymbol{E}\boldsymbol{V}\boldsymbol{j}_l = 0 \tag{166}$$

Note that in this specific case Eaves uses a stricter definition of equilibrium where we need to have exact market clearance (i.e. excess demand is equal to 0) as opposed to the classical definition of market clearance provided by Arrow-Debreu, which allows 0 prices if excess demand is negative for any strictly positive price (i.e. the good is not demanded by any agent). That is, Eaves is looking for only strictly positive prices to the above equation. This is important since this means that equilibrium prices might not always exists (i.e., the Arrow-Debreu theorems' equilibrium existence proofs are not constructed for strictly positive prices).

Solving for prices for the above system of equations is equivalent to solving for prices in the following system:

$$(D(\boldsymbol{p})\boldsymbol{V}^T\boldsymbol{E})^T\boldsymbol{j}_l - (D(\boldsymbol{p})\boldsymbol{V}^T\boldsymbol{E})\boldsymbol{j}_l = 0 \tag{167}$$

This system is also equivalent to the following system:

$$(E - D(\boldsymbol{V}^T\boldsymbol{E}\boldsymbol{j}_l))^T\boldsymbol{p} = 0 \tag{168}$$

As previously mentioned, solving this equation alone is not enough since strictly positive prices might not exist. We need to also discuss existence of strictly positive prices. The above equation tells us that the existence (and uniqueness) of equilibrium prices is solely based on the matrix $\boldsymbol{V}^T\boldsymbol{E}$ and not only on its individual components.

Definition 7.4. *A matrix is **line-sum-symmetric** iff its corresponding row and column sums are equal.*

Namely, observing equation (167), we can see that in order for strictly positive equilbrium prices to exist, we need to prove the existence of prices $D(\boldsymbol{p})$ such that $D(\boldsymbol{p})\boldsymbol{V}^T\boldsymbol{E}$ is line-sum-symmetric. In the words of Eaves, "solving for prices is the task of finding a positive row-scaling $\boldsymbol{p}$, of $\boldsymbol{V}^T\boldsymbol{E}$ which is line-sum-symmetric.

In order to describe the necessary and sufficient conditions for the existence of strictly positive prices, we need to introduce one more concept.

Definition 7.5. *Given two goods i and j, good i is defined to **access** good j iff $i = j$ or if there is a sequence of goods $i = g_1, g_2, \dots, g_r, j = g_{r+1}$ such that $(\boldsymbol{V}^T\boldsymbol{E})_{g_k, g_{k+1}} > 0$ for $k = 1, \dots, r$.*

In other words, a good i accesses good j if there is a sequence of $t_1, \dots, t_r$ such that agent t_k possesses good k and desires good g_{k+1} for $k = 1, \dots, r$.

Definition 7.6. *The matrix $\boldsymbol{V}^T\boldsymbol{E}$ is defined to have symmetric access if for every pair of goods i and j they access each other or neither accesses the other.*

Definition 7.7. *The matrix $\boldsymbol{V}^T\boldsymbol{E}$ is defined to have full access iff for every pair of goods i and j, i and j access each other.*

Using all these definitions, we now present a theorem proven by Eaves in an earlier paper:

Theorem 7.8. *A square non-negative matrix $\boldsymbol{V}^T\boldsymbol{E}$ has a line-sum-symmetric positive row scaling iff $\boldsymbol{V}^T\boldsymbol{E}$ has symmetric access. A square non-negative matrix $\boldsymbol{V}^T\boldsymbol{E}$ has a line-sum-symmetric positive row scaling that is unique iff $\boldsymbol{V}^T\boldsymbol{E}$ has full access.*

This shows that strictly positive equilibrium prices exist iff $\boldsymbol{V}^T\boldsymbol{E}$ has symmetric access. This brings us to the next theorem which we do not prove as it is relatively complicated.

Theorem 7.9. *The Cobb-Douglas Arrow-Debreu exchange market $(\boldsymbol{V}, \boldsymbol{E})$ has unique strictly positive prices iff $\boldsymbol{V}^T\boldsymbol{E}$ has symmetric access.*

Before getting to the computation of the equilibrium we discuss the concept of submarkets.

Definition 7.10. *A **submarket** is a subset of agents and goods such that any agent possesses and desires only goods in the submarket and does not own or desire goods outside of the submarket.*

Note that every good in a submarket has full access.

It turns out that we can use a very fast decomposition algorithm (that runs in $O(m^2)$ time for a square matrix of size $m \times m$) on the matrix $\boldsymbol{V}^T\boldsymbol{E}$ such that we can obtain M independent submarkets $(\boldsymbol{B}_k, \boldsymbol{W}_k)$ for $k = 1, \dots, M$. Since these markets are submarkets, for $k = 1, \dots, M$, $\boldsymbol{B}_k^T\boldsymbol{W}_k$ has a unique row scaling. This means that we can use Gaussian elimination that runs in $O(m^3)$ time for a square matrix of size $m \times m$ to obtain the prices of the goods in each submarket and then combine the prices in each submarket and normalize them to obtain the equilbrium prices for the Arrow-Debreu market. This approach takes a running time of $O(m^3)$ for a matrix $\boldsymbol{V}^T\boldsymbol{E}$ of size $m \times m$.

7.4.2 The Tatonnement Process

The Tatonnement process (from French “Trial and error”) is a process guaranteed to converge to equilibrium prices allocations for a class of utility functions called **Gross Substitutes**. The Tatonnement process (also called the Walrassian auction) was invented way before the Arrow-Debreu model. Briefly, it is an auction that adjusts the prices of the goods based on the excess demand of goods.

Define the following optimization program whose output is the utility maximizing and budget constrained consumption bundle of consumer i (i.e., the demand of consumer i):

$$D^i(\boldsymbol{p}) = \left\{ \begin{array}{c} \arg\max_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i) \\ \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p} + \sum_{j=1}^{n} \alpha_{ij} \boldsymbol{y}_j \cdot \boldsymbol{p} \\ \boldsymbol{x}_i \in X_i \end{array} \right\} \tag{169}$$

where $\boldsymbol{y}_j$ is chosen arbitrarily.

Define the excess demand function $z : P \to P$:

$$z(\boldsymbol{p}) = \sum_{i \in [m]} D^i(\boldsymbol{p}) - \sum_{i \in [m]} \boldsymbol{e}_i \tag{170}$$

We now define the Gross Substitutes condition within the context of the Arrow-Debreu model (which is different than the definition of the gross substitutes condition for auctions with indivisible items but these definitions are related). In the context of the Arrow-Debreu model, the Gross Substitute condition is a characteristic of the excess demand function rather than that of the utility functions.

Gross Substitutes: Let $\boldsymbol{p}$ and $\boldsymbol{p}^{'}$ bet two different price vectors such that $\forall h \in [l], p_h \leq p_h^{'}$ and $\exists k \in [l]$ such that $p_k < p_k^{'}$. Then an excess demand function z fulfills the Gross Substitutes condition iff:

$$\forall h \neq k, \;\; z_h(\boldsymbol{p}) > z_k(\boldsymbol{p}^{'}) \tag{171}$$

where z_h denotes the h^{th} coordinate in the output vector of the excess demand function (i.e., the excess demand for the commodity h) Equivalently in calculus terms, we can state the gross substitutes condition as:

$$\forall h \neq k, \;\; \frac{\partial z_h}{\partial p_k} > 0 \tag{172}$$

In other words, if the prices of some goods are increased while the prices of some other goods are held fixed, this can only cause an increase in the demand of the goods whose price stayed fixed.

Before introducing the Tatonnement process we will introduce two more conditions that hold under the Arrow-Debreu Theorem Assumptions.

Homogeneity: $\forall \alpha > 0, z(\alpha \boldsymbol{p}) = z(\boldsymbol{p})$.
Proof:

We will first show that multiplying prices by a strictly positive scalar does not change demand.

$$D^i(\alpha \boldsymbol{p}) = \left\{ \begin{array}{c} \arg\max_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i) \\ \boldsymbol{x}_i \cdot \alpha \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \alpha \boldsymbol{p} \\ \boldsymbol{x}_i \in X_i \end{array} \right\} \tag{173}$$

$$= \left\{ \begin{array}{c} \arg\max_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i) \\ \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p} \\ \boldsymbol{x}_i \in X_i \end{array} \right\} \tag{174}$$

$$= D^i(\boldsymbol{p}) \tag{175}$$

$$z(\alpha \boldsymbol{p}) = \sum_{i \in [m]} D^i(\alpha \boldsymbol{p}) - \sum_{i \in [m]} \boldsymbol{e}_i \tag{176}$$

$$= \sum_{i \in [m]} D^i(\boldsymbol{p}) - \sum_{i \in [m]} \boldsymbol{e}_i \tag{177}$$

$$= z(\alpha \boldsymbol{p}) \tag{178}$$

Walras' Law: $\boldsymbol{p} \cdot z(\boldsymbol{p}) = 0$, meaning that the total spending and total income in the economy are equal to each other.
Proof:

$$\boldsymbol{p} \cdot z(\boldsymbol{p}) = \boldsymbol{p} \cdot \left(\sum_{i \in [m]} D^i(\boldsymbol{p}) - \sum_{i \in [m]} \boldsymbol{e}_i \right) \tag{179}$$

$$= \boldsymbol{p} \cdot \sum_{i \in [m]} D^i(\boldsymbol{p}) - \boldsymbol{p} \cdot \sum_{i \in [m]} \boldsymbol{e}_i \tag{180}$$

$$\tag{181}$$

Remember from the Arrow-Debreu Theorem conditions that the utility of the agents are non satiated and quasi-concave. An implication of this is that, in order for the agent to maximize their utilities, they have to spend their entire budget. That is, agent's spending is equal to the value of the bundle Mathematically, this gives:

$$\boldsymbol{p} \cdot \sum_{i \in [m]} D^i(\boldsymbol{p}) = \boldsymbol{p} \cdot \sum_{i \in [m]} \boldsymbol{e}_i \tag{182}$$

Hence, going back to the original problem we get:

$$\boldsymbol{p} \cdot z(\boldsymbol{p}) = \boldsymbol{p} \cdot \sum_{i \in [m]} D^i(\boldsymbol{p}) - \boldsymbol{p} \cdot \sum_{i \in [m]} \boldsymbol{e}_i \tag{183}$$

$$= \boldsymbol{p} \cdot \sum_{i \in [m]} \boldsymbol{e}_i - \boldsymbol{p} \cdot \sum_{i \in [m]} \boldsymbol{e}_i \tag{184}$$

$$= 0 \tag{185}$$

We now introduce the Tatonnement process:

Tatonnement Process: Let $G(.)$ be a monotonous sign preserving function. The Tatonement is a time process, which at each time step t changes prices in the following manner:

$$\frac{dp_h}{dt} = G(z_h(\boldsymbol{p})) \tag{186}$$

In words, the Tatonnement process, increases the prices of goods that are demanded in excess and decreases the prices of the goods are supplied in excess at each time step.

Theorem 7.11. *If the excess demand function satisfies the homogeneity, Walras' Law and Gross Substitutes conditions, then the Tatonnement process converges to the equilibrium prices* $\mathbf{p}^*$ *for the Arrow-Debreu Model.*

Note that these conditions are **necessary** for the Tatonnement process to converge to equilibrium prices but we will only prove that they are sufficient.

Proof:

To prove this result, we introduce a theorem proved by Kenneth Arrow and Leonid Hurwicz in the 60s [3]. As the proof is relatively involved, we skip it.

Theorem 7.12. ***Weak Axiom of Revealed Preferences***

If the excess demand z *satisfies the homogeneity, Walras' law, gross substitutes conditions, then for every non-equilibrium price* $\mathbf{p}$ *and equilibrium price* $\mathbf{p}^*$*, we have:*

$$\sum_{h \in [l]} p_h^* z_h(\boldsymbol{p}) > 0 \tag{187}$$

We will now establish the convergence of the Tatonnement Process using a Lyapunov potential function, a method used in Dynamic Systems to establish the convergence of time processes to equilibria. We define the following potential function:

$$V(\boldsymbol{p}) = \frac{1}{2} \sum_{h \in [l]} (p_h - p_h^*)^2 \tag{188}$$

The idea behind the potential function is to calculate the distance between the price vector calculated by the Tatonnement process at any time step and the equilibrium price vector. Then, if we can show that for each successive price vector obtained by the Tatonnement process this distance decreases we essentially have proven that asymptotically the process needs to output the equilibrium price vector. To do so, we now parametrize the price vector outputted by the tatonnement process at each time step t by t. We then get the following potential function:

$$V(\boldsymbol{p}) = \frac{1}{2} \sum_{h \in [l]} (p_h(t) - p_h^*)^2 \tag{189}$$

Taking the derivative of this potential function with respect to t, we get:

$$\frac{dV}{dt} = \sum_{h \in [l]} (p_h(t) - p_h^*) \frac{dp_h}{dt} \tag{190}$$

By the definition of the Tatonnement process we have:

$$\frac{dp_h}{dt} = G(z_h(\boldsymbol{p})) \tag{191}$$

If we pick $G(\boldsymbol{p}) = \boldsymbol{p}$ (which is a monotonous sign preserving function) and we substitute it into the derivative of the potential function, we get:

$$\begin{aligned} \frac{dV}{dt} &= \sum_{h \in [l]} (p_h(t) - p_h^*) z_h(p(t)) && (192) \\ &= \sum_{h \in [l]} p_h(t) z_h(p(t)) - \sum_{h \in [l]} p_h^* z_h(p(t)) && (193) \\ & && (194) \end{aligned}$$

By Walras' law, we know that $\sum_{h \in [l]} p_h(t) z_h(p(t)) = 0$ and by the weak axiom of revealed preferences, we know that $p_h^* z_h(p(t)) > 0$. We then get:

$$\begin{aligned} \frac{dV}{dt} &= \sum_{h \in [l]} p_h(t) z_h(\boldsymbol{p}(t)) - \sum_{h \in [l]} p_h^* z_h(\boldsymbol{p}(t)) && (195) \\ &= -\sum_{h \in [l]} p_h^* z_h(\boldsymbol{p}(t)) && (196) \\ &< 0 && (197) \end{aligned}$$

The derivative of the potential function with respect to each time step t being negative implies that at each iteration of the Tatonnement process the distance between the price vector outputted by the Tatonnement process and the equilibrium price vector only decreases which means that as t goes to infinity the Tatonnement process's output price vector converges to the equilibrium price vector.

8 Graphical Economy Model

8.1 Model

A **graphical economy** consists of:

1. A set of divisible goods $[l]$
2. A set of agents $[m]$ embedded in some undirected graph $G = \{[m], E\}$. For notational clarity, we will use the reflexive symmetric binary relation $i \simeq j$ to denote an edge between agents $i, j \in [m]$. We will also use the notation $i \sim j$ to mean $i \simeq j$ and $i \neq j$.
3. Each agent $i \in [m]$ is characterized by:
   - A utility function $u_i : \mathbb{R}^l \to$ '$\mathbb{R}_+$ which denotes the preference of the agent i over the space of consumption bundles $\mathbb{R}^l$
   - An endowment $\boldsymbol{e}_i \in \mathbb{R}^l_+$ where e_{ih} denotes the amount of good $h \in [l]$ i is endowed with.

An instance of a graphical economy is given by a tuple $(l, G, U, \boldsymbol{E})$. When clear from context, we simply write $(G, U, \boldsymbol{E})$

8.2 Model Outcome

Goods are assigned **local prices** $\boldsymbol{P} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m)^T \in \mathbb{R}^l_+$ such that $p_{ih} \geq 0$ denotes the price agent $i \in [m]$ *charges* for good h. A **consumption plan**[13] $\boldsymbol{X}_i \in \mathbb{R}^{l \times m}_+$ is a map from buyers to goods, represented as a matrix such that $x_{ijh} \geq 0$ denotes the amount of good $h \in [l]$ purchased by agent $i \in [m]$ from trade partner $j \in [m]$. To enforce the condition that agent only trade with their trade partner determined by the graph underlying the economy, we set $\boldsymbol{x}_{ij} = 0$ for all $i \not\simeq j$. An **outcome** of the graphical economy model is a tuple of consumption plans and local prices, i.e., $(\boldsymbol{X}, \boldsymbol{P}) \in \mathbb{R}^{m \times m \times l} \times \mathbb{R}^{m \times m}$.

An **KKO (Kakade, Kearns, Ortiz) equilibrium** of a graphical economy $(G, U, \boldsymbol{E})$ is an outcome $(\boldsymbol{X}^*, \boldsymbol{P}^*)$ such that:

- (Utility Maximization) Agents maximize their utility constrained by their budget:

$$\left(\sum_{j \simeq i} \boldsymbol{x}^*_{ij}\right) \in \underset{\boldsymbol{x}_{ij} \in \mathbb{R}^l_+ : \sum_{j \simeq i} \boldsymbol{x}_{ij} \cdot \boldsymbol{p}^*_j \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^*_i}{\arg\max} u_i \left(\sum_{j \simeq i} \boldsymbol{x}_{ij}\right) \qquad \forall i \in [m] \tag{198}$$

- (Local Clearance)

$$\sum_{j \simeq i} \boldsymbol{x}^*_{ji} = \boldsymbol{e}_i \qquad \forall i \in [m] \tag{199}$$

[13]Note that in the traditional Arrow-Debreu economy setting, the consumption plan is referred to as an allocation, however since we will consider agent who purchase goods for both consumption and reselling we will be using different terminology.

Theorem 8.1 (KKO Existence Theorem)**.** *Suppose that the following conditions are satisfied:*

1. u_i *is continuous*
2. u_i *is quasi-concave, i.e.,* $\forall \lambda \in (0,1),\ \ u_i(\lambda x + (1-\lambda)y) \geq \min\{u_i(x), u_i(y)\}$
3. u_i *is locally non-satiated, i.e.,* $\forall \boldsymbol{x} \in \mathbb{R}^l_+,\ \ \forall \epsilon > 0,\ \ \exists \boldsymbol{y} \in B_\epsilon(\boldsymbol{x}), u_i(\boldsymbol{y}) > u_i(\boldsymbol{x})$
4. *Each agent is endowed with a strictly positive amount of all commodities, i.e.,* $\forall i \in [m], [l] \in [l] e_{ih} > 0$.

Then an equilibrium outcome of the Arrow-Debreu Model exists.

We note that in a graphical economy, agents are limited to trade with agents only one hop away from them, which is an unrealistic presentation of real world behavior. A solution to this is provided by Andrade et al. who introduce graphical economies with resale.

9 Graphical Economy with Resale Model

9.1 Model

A **graphical economy with resale** consists of:

1. A set of divisible goods $[l]$
2. A set of agents $[m]$ embedded in some undirected graph $G = \{[m], E\}$. For notational clarity, we will use the reflexive symmetric binary relation $i \simeq j$ to denote an edge between agents $i, j \in [m]$. We will also use the notation $i \sim j$ to mean $i \simeq j$ and $i \neq j$.
3. Each agent $i \in [m]$ is characterized by:
   - A utility function $u_i : \mathbb{R}^l \to$ '$\mathbb{R}_+$ which denotes the preference of the agent i over the space of consumption bundles $\mathbb{R}^l$
   - An endowment $\boldsymbol{e}_i \in \mathbb{R}^l_+$ where e_{ih} denotes the amount of good $h \in [l]$ i is endowed with.
   - A resale bound $b_i \in \mathbb{R}_+$ which represents the maximum value of the bundle which the agent is willing to resell.

An instance of a graphical economy is given by a tuple $(l, G, U, \boldsymbol{E})$. When clear from context, we simply write $(G, U, \boldsymbol{E})$

9.2 Model Outcome

Goods are assigned **local prices** $\boldsymbol{P} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m)^T \in \mathbb{R}^l_+$ such that $p_{ih} \geq 0$ denotes the price agent $i \in [m]$ *charges* for good h. A **consumption plan** $\boldsymbol{X}_i \in \mathbb{R}^{l \times m}_+$ is a map from buyers to goods, represented as a matrix such that $x_{ijh} \geq 0$ denotes the amount of good $h \in [l]$ purchased by agent $i \in [m]$ from trade partner $j \in [m]$. A **resale plan** $\boldsymbol{y}_i \in \mathbb{R}^{l \times m}_+$ is a map from buyers to goods, represented as a matrix such that $x_{ijh} \geq 0$ denotes the amount of good $h \in [l]$ purchased by agent $i \in [m]$ from trade partner $j \in [m]$ *for the purposes of resale.* To enforce the condition that agent only trade with their trade partner determined by the graph underlying the economy, we set $\boldsymbol{x}_{ij} = 0$ for all $i \not\simeq j$. An **outcome** of the graphical economy model is a tuple of consumption plans and local prices, i.e., $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{P}) \in \mathbb{R}^{m \times m \times l} \times \mathbb{R}^{m \times m \times l} \times \mathbb{R}^{m \times m}$.

An **resale equilibrium** of a graphical economy $(G, U, \boldsymbol{E})$ is an outcome $(\boldsymbol{X}^*, \boldsymbol{Y}^*, \boldsymbol{P}^*)$ such that:

- (Utility Maximization) Agents maximize their utility constrained by their budget:

$$\left(\sum_{j \simeq i} \boldsymbol{x}^*_{ij}\right) \in \underset{\boldsymbol{x}_{ij} \in \mathbb{R}^l_+ : \sum_{j \simeq i} \boldsymbol{x}_{ij} \cdot \boldsymbol{p}^*_j \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^*_i + \sum_{j \simeq i} \left(\boldsymbol{p}^*_i - \boldsymbol{p}^*_j\right) \cdot \boldsymbol{y}^*_{ij}}{\arg\max} u_i(\boldsymbol{x}) \qquad \forall i \in [m] \tag{200}$$

- (Optimal Arbitrage) The agents purchase the profit maximizine resale plan:

$$\boldsymbol{y}^*_{ij} \in \underset{\boldsymbol{Y} \in \mathbb{R}^l_+ : \sum_{j \simeq i} \boldsymbol{y}_{ij} \cdot \boldsymbol{p}^*_j \leq b_i}{\arg\max} \sum_{j \simeq i} \left(\boldsymbol{p}^*_i - \boldsymbol{p}^*_j\right) \cdot \boldsymbol{y}_{ij} \qquad \forall i \in [m] \tag{201}$$

- (Local Clearance) The quantities of goods purchased for resale and consumption are equal to the sum of the quantity of goods sold for resale and the consumers' endowment locally.

$$\sum_{j \simeq i} \boldsymbol{x}^*_{ji} + \sum_{j \simeq i} \boldsymbol{y}^*_{ji} = \boldsymbol{e}_i + \sum_{j \simeq i} \boldsymbol{y}^*_{ij} \qquad \forall i \in [m] \tag{202}$$

Theorem 9.1 (Resale Existence Theorem)**.** *Suppose that the following conditions are satisfied:*

1. u_i *is continuous*
2. u_i *is quasi-concave, i.e.,* $\forall \lambda \in (0, 1), \quad u_i(\lambda x + (1 - \lambda) y) \geq \min \{u_i(x), u_i(y)\}$
3. u_i *is strictly increasing in one good*
4. *Each agent is endowed with a strictly positive amount of all commodities, i.e.,* $\forall i \in [m], [l] \in [l] e_{ih} > 0$ *or has a strictly postitive resale bound* $\forall i \in [m], b_i > 0$.
5. *For every good* $h \in [l]$*, there exists an agent* $i \in i$ *such that* $e_{ih} > 0$

Then an equilibrium outcome of the Arrow-Debreu Model exists.

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