Modelling Markets

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List of things to do:

1. Change text about tatonnement being only process converging, explain importance of tatonnement better
2. citations for AD, Fisher, PM
3. add text on elasticity of substitution and its meaning
4. Define perfect substitutes, weak complements etc...
5. fix eisenberg-gale program text to say that it works for continuous concave utility functions of homogeneous degree function
6. Write english for arrow-debreu by telling what they are
7. Schmyrev's program fix it
8. fix arrow debreu excahnge thm two production set to consumption set
9. fix warp so eqm price is different then price and then check for axiom
10. add gross substitutes (i.e. all classes of utility functions in the utility section)
11. Modify Fisher so that the number of goods is not 1 throughout
12. remove arrow-debreu parameter preference and change letter to match fisher
13. add commas to subscript
14. make fisher AD proof in 3 parts
15. make sure to add to reduction of fisher AD that p_{m+1} is strictly positive.
16. add checking equilibrium prices for linear case (i.e. maxflow computation) and then

1 Pari-Mutuel Model

In 1867, Spanish entrepreneur Joseph Oller invented parimutuel betting, a form of wagering that is still popular today and handles billions of waged dollars every year! The pari-mutuel model is a system of betting in which all bets are placed on a set of possible exlusive outcomes. Once the outcome is determined bettor who bet on the winning outcome split the total amount wagered in proportion to the size of their wagers. To explain this system better we will consider the application of pari-mutuel betting to betting in horse races.

In this setting, each bettor bets on a horse. The house collects the bets and pays the participants that bet on the winning horse, the total amount of money collected multiplied by the proportion of each winner's bet in the total amount bet on the winning horse.

The first principle of the pari-mutuel model is that you should not try to bet on the horse that has the highest chance of winning, but on the horse that has the best pay out ratio relative to your belief of its chance of winning. The reason for this is simply that as more people bet on the same horse the likelihood of any bettor making a profit goes to 0. To better illustrate this principle, consider the trivial case when every agent bets all their money on the same horse. If that horse ends up winning, every bettor gets back the amount of money he/she bet but makes no profit whatsoever. Since, the goal of betting is to make a profit, a bettor is better off guessing a winning horse on which only a small amount of money is bet.

One of the main issues

1.1 Model

We now introduce the mathematical model proposed by Edmund Eisenberg and David Gale:[1]

- m bettors labelled $B_1, \ldots, B_m$
- Each bettor $B_i, \forall i \in [m]$, has a budget b_i. We assume that the sum of the budgets is equal to one, i.e., $\sum_{i=1}^{m} b_i = 1$
- n horses labelled, $H_1, \ldots, H_n$
- Matrix of *subjective opinions* of winning, where each entry $(i, j)^{th}$ is the prior that each bettor B_i has on the probability of winning of horse H_j such that $\boldsymbol{P} = [p_{ij}]_{i \in [m], j \in [n]}$
- WLOG, we assume that each horse i has at least one subjective opinion of winning that is strictly positive, i.e. $p_{ij} > 0$. Otherwise, we could simply not consider that horse in the betting process because no one will bet money on it (since no bettor believes that the horse will win).

1.2 Outcome of the Model

Each bettor B_i bets part of his budget on any horse H_j. Let $\boldsymbol{\beta} = [\beta_{ij}]_{i \in [m], j \in [n]}$ be the **bet allocation** matrix describing the amount that bettor B_i bet on horse H_j. After the bets have been made, it is possible to determine an aggregation of all bettors' opinions to determine the aggregated winning probability of each horse. Let $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)^T$ be the **final track probabilities** vector which represents the probability of each horse winning based on the bets have been made.[2] A Pari-Mutuel outcome is a tuple $(\boldsymbol{\beta}, \boldsymbol{\pi})$ consisting of bet allocations and final track probabilities.

Any Pari-Mutuel outcome must satisfy the **budget relation**: $\sum_{j=1}^{m} \beta_{ij} \leq b_i$ (i.e., the sum of the bets of each bettor is equal to their budget) and the **Pari-Mutuel condition**: $\sum_i \beta_{ij} = \pi_j$ (i.e., the sum of the bets on each horse is equal to their final track probability). The Pari-Mutuel condition simply arithmetizes the main principle behind the model which is that the probability of a horse winning is proportional to the amont bet on that horse. Note that, since we assumed that the sum of all of the budgets is equal to 1, in this case the proportonality constant is exactly 1.

[1]Fun fact: Gale was a Professor in the applied math department here at Brown!

[2]This vector of probability represents the aggregation of all bettors subjective probabilities into a unique probability vector.

As stated before, the goal of each bettor is to bet on the horse that has the best pay out ratio relative to their belief of the horse's winning odds. Since the bettors make a bigger profit when the winning horse has less money bet on it, the goal of each bettor can be modelled as betting on the horses with the highest ratio of subjective probability of winning to final track probability[3]. A Pari-Mutuel outcome $(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*)$ in which the bettors bet in this manner is said to be **expectation maximizing**. We arithmetize the expectation maximization condition as follows:

$$\text{if } \mu_i = \max_s \frac{p_{is}}{\pi_s^*} \text{ and } \beta_{ij}^* > 0, \text{ then } \mu_i = \frac{p_{ij}}{\pi_j^*} \tag{1}$$

A Pari-Mutuel outcome is **optimal** iff bettors expectation maximizing and the outcome respects the budget relation as well as the Pari-Mutuel condition.

Now, observe that the final track probabilities cannot be determined before the bettors have put their money on their horses. However, bettors can also not maximize their expectation without knowing the final track probabilities! So one question to ask is whether if an optimal outcome of the Pari-Mutuel model exists at all!

1.3 Optimal Solution

We now introduce the Eisenberg-Gale convex program.

$$\max_{\boldsymbol{\xi}} \sum_{i=1}^{m} b_i \log\left(\sum_{j=1}^{n} p_{ij}\xi_{ij}\right) \tag{2}$$

$$\xi_{ij} \geq 0 \tag{3}$$

$$\sum_{i=1}^{m} \xi_{ij} = 1 \tag{4}$$

The optimal output $\bar{\boldsymbol{\xi}} = \left[\bar{\xi}_{ij}\right]_{i\in[m],j\in[n]}$ to this convex program can be used to calculate the optimal allocation $(\boldsymbol{\beta}, \boldsymbol{\pi})$. This is also a proof of the existence an optimal allocation for the Pari-Mutuel problem (given that our claim in the previous sentence is correct) since any convex program is guaranteed to have a minimum.

Theorem 1.1. *Let $\bar{\boldsymbol{\xi}}$ be a solution of the Eisenberg-Gale program. An optimal allocation $(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*)$ for the Pari-Mutuel market can be calculated as:*

$$\pi_j^* = \max_i \frac{b_i p_{ij}}{\sum\limits_{s=1}^{n} p_{is}\bar{\xi}_{is}} \tag{5}$$

$$\beta_{ij}^* = \bar{\xi}_{ij}\pi_j^* \tag{6}$$

Proof: First, we claim the following, which we use in the rest of the proof:

$$\text{if } \bar{\xi}_{ij} > 0 \text{ then } \pi_j = \frac{b_i p_{ij}}{\sum\limits_{s=1}^{n} p_{is}\bar{\xi}_{is}} \tag{7}$$

To see this, suppose that this claim is false and that for some i, j, $\bar{\xi}_{ij} > 0$ and that $\pi_j > \frac{b_i p_{ij}}{\sum_{s=1}^{n} p_{is}\bar{\xi}_{is}}$. By definition of π_j, we then have a $\bar{\xi}_{kj}$ for some k which gives $\pi_j = \frac{b_k p_{kj}}{\sum_{s=1}^{n} p_{ks}\bar{\xi}_{ks}} > \frac{b_i p_{ij}}{\sum_{s=1}^{n} p_{is}\bar{\xi}_{is}}$. This then implies that we could decrease $\bar{\xi}_{ij}$

[3]Remember that a lower final track probability means less money bet on a horse due to the Pari-Mutuel condition

and increase $\bar{\xi}_{kj}$ to increase the objective function of the Eisenberg-Gale program. This, however, is a contradiction since $\bar{\xi}$ maximizes the Eisenberg-Gale program. In other words, this fact simply comes from the fact that $\bar{\xi}$ maximizes the Eisenberg-Gale program.

Budget constraint condition: Combining conditions (6) and (7), we get:

$$\beta^*_{ij} = \bar{\xi}_{ij}\pi^*_j \tag{8}$$

$$= \bar{\xi}_{ij}\frac{b_i p_{ij}}{\sum\limits_{s=1}^{n} p_{is}\bar{\xi}_{is}} \tag{9}$$

$$= b_i \frac{p_{ij}\bar{\xi}_{ij}}{\sum\limits_{s=1}^{n} p_{is}\bar{\xi}_{is}} \tag{10}$$

Summing the above on j, we get: $\sum_{j=1}^{m} \beta^*_{ij} = b_i \frac{\sum\limits_{j=1}^{m} p_{ij}\bar{\xi}_{ij}}{\sum\limits_{s=1}^{n} p_{is}\bar{\xi}_{is}} = b_i$. This confirms the budget constraint condition.

Pari-Mutuel Condition: By constraint (4), we know that $\sum_{i=1}^{m} \bar{\xi}_{ij} = 1$. Hence, summing up β^*_{ij} on i we recover the final track probabilities, i.e., $\sum_{i=1}^{m} \beta^*_{ij} = \sum_{i=1}^{m} \bar{\xi}_{ij}\pi^*_j = \pi^*_j$. This confirms that the allocation respects the Pari-Mutuel condition.

Expectation Maximization

From (5), we know that:

$$\pi^*_j = \max_i \frac{b_i p_{ij}}{\sum\limits_{s=1}^{n} p_{is}\bar{\xi}_{is}} \tag{11}$$

$$\frac{1}{\pi_j} = \min_i \frac{\sum\limits_{s=1}^{n} p_{is}\bar{\xi}_{is}}{b_i p_{ij}} \tag{12}$$

$$\frac{p_{ij}}{\pi_j} = \min_i \frac{\sum\limits_{s=1}^{n} p_{is}\bar{\xi}_{is}}{b_i} \tag{13}$$

$$\frac{p_{ij}}{\pi_j} \leq \frac{\sum\limits_{s=1}^{n} p_{is}\bar{\xi}_{is}}{b_i} \tag{14}$$

Recall from (1) that:

$$\mu_i = \max_s \frac{p_{is}}{\pi^*_s} \tag{15}$$

Substituting π^*_s with the expression from (5), we get:

$$\mu_i = \max_s p_{is} \frac{\sum\limits_{k=1}^{n} p_{ik}\bar{\xi}_{ik}}{b_i p_{is}} \tag{16}$$

$$= \max_s \frac{\sum\limits_{k=1}^{n} p_{ik}\bar{\xi}_{ik}}{b_i} \qquad = \frac{\sum\limits_{k=1}^{n} p_{ik}\bar{\xi}_{ik}}{b_i} \tag{17}$$

Since we assumed that there exists at least one entry in each column of $\boldsymbol{P}$ that is strictly positive, we know that each π_j is positive[4] Then, combining facts (6), (17), (14) and (7), we get exactly (1).

[4]Observe the objective function of the Eisenberg-Gale program to convince yourself about this fact.

2 Utilities

2.1 Preference Relationships (Ordinal Utilities) & Cardinal Utilities

An important feature of economic markets is that they aggregate the preferences of consumers to what we know as prices. As a result, in order to define clear market model, we have to understand clearly the theory of utility functions.

Suppose an agent chooses from a set of goods $G = \{1, 2, 3, \ldots\}$. For example, one can think of these goods as different TV sets or cars. Given two goods, $x \in G$ and $y \in G$:

- the agent weakly prefers x over y if x is at least as good as y. To avoid us having to write "weakly prefers" repeatedly, we simply write $x \succcurlyeq y$.
- the agent strongly prefers x over y if x is better than y. To avoid us having to write "strongly prefers" repeatedly, we simply write $x \succ y$.

We now put some basic structure on the agent's preferences by adopting two axioms.

Completeness Axiom: For every pair $x, y \in X$, either $x \succcurlyeq y, y \succcurlyeq x$, or both.

Transitivity Axiom: For every triple $x, y, z \in X$, if $x \succcurlyeq y$ and $y \succcurlyeq z$ then $x \succcurlyeq z$

An agent has **complete preferences** if they can compare any two objects. An agent has **transitive preferences** if their preferences are internally consistent.

While it is natural to think about preferences, it is often more convenient to associate different numbers to different goods, and have the agent choose the good with the highest number. These numbers are called utilities. In turn, a utility function tells us the utility associated with each good $x \in X$, and is denoted by $u(x) \in \Re$. We say a utility function $u(x)$ represents an agent's preferences if $u(x) \geq u(y)$ if and only if $x \succcurlyeq y$. This means than an agent makes the same choices whether they uses their preference relation, $\succcurlyeq$ or her utility function $u(x)$.

Theorem 2.1. ***Utility Representation Theorem*** *Suppose the agent's preferences, $\succcurlyeq$, are complete and transitive, and that the set of goods G is finite. Then there exists a utility function $u(x) : G \to \mathbb{R}$ which represents $\succcurlyeq$*

Proof: For any good x, let $NBT(x) = \{y \in X | x \succcurlyeq y\}$ be the goods that are "no better than" x. The utility of x is simply given by the number of items in $NBT(x)$. That is

$$u(x) = |NBT(x)| \tag{18}$$

We now verify that the construction we have given is valid. Suppose $x \succcurlyeq y$. Pick any $z \in NBT(y)$ by the definition of $NBT(y)$, we have $y \succcurlyeq z$ since preferences are complete, we know that z is comparable to x. Transitivity then tells us that $x \succcurlyeq z$, so $z \in NBT(x)$. We have therefore shown that every element of $NBT(y)$ is also an element of $NBT(x)$, that is, $NBT(y) \subseteq NBT(x)$. As a result,

$$u(x) = |NBT(x)| \geq |NBT(y)| = u(y) \tag{19}$$

which confirms our claim.

Theorem 2.2. *Suppose $u(x)$ represents the agent's preferences, $\succcurlyeq$, and $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function. Then the new utility function $v(x) = f(u(x))$ also represents the agent's preferences $\succcurlyeq$.*

Proof: The proof is simply a rewriting of definitions. Suppose $u(x)$ represents the agent's preferences. If $x \succcurlyeq y$ then $u(x) \geq u(y)$ and $f(u(x)) \geq f(u(y))$, so that $v(x) \geq v(y)$. Conversely, if $v(x) \geq v(y)$ then, since $f(.)$ is strictly increasing, $u(x) \geq u(y)$ and $x \succcurlyeq y$. Hence $v(x) \geq v(y) \quad$ if and only if $\quad x \succcurlyeq y$ and $v(x)$ represents $\succcurlyeq$.

Note that the two theorems we presented hold also for infinitely big sets of goods (or a continuum of goods), but we will not go over this, as computational aspects of markets rely on a finite sets of goods.

2.2 Properties of Preferences

Monotonicity Preferences are monotone if for any two bundles $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$

$$x_i \geq y_i \text{ for each } i \tag{20}$$

$$x_i > y_i \text{ for some } i \tag{21}$$

implies $\boldsymbol{x} \succ \boldsymbol{y}$ In words, preferences are monotone if more of any good makes the agent strictly better off. While monotonicity is stated in terms of preferences, we can rewrite it in terms of utilities.

Preferences are monotone if for any two bundles $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$x_i \geq y_i \text{ for each } i \tag{22}$$

$$x_i > y_i \text{ for some } i \tag{23}$$

implies $u(\boldsymbol{x}) > u(\boldsymbol{y})$.

Non-Satiation Let G be the set of goods. A preference relation is non-satiated:

$$\forall \boldsymbol{x} \in \mathbb{R}^{|G|}, \;\; \epsilon > 0, \;\; \exists \boldsymbol{y} \in \left\{ \boldsymbol{y} | \boldsymbol{y} \in \mathbb{R}^{|G|}, ||\boldsymbol{y} - \boldsymbol{x}|| \leq \epsilon \right\}, \quad \boldsymbol{y} \succ \boldsymbol{x} \tag{24}$$

In words, for any bundle of goods, there exists an arbitrarily close bundle of goods that is preferred. While non-satiation is stated in terms of preferences, we can rewrite it in terms of utilities. Let G be the set of goods. A preference relation is non-satiated:

$$\forall \boldsymbol{x} \in \mathbb{R}^{|G|}, \;\; \epsilon > 0, \;\; \exists \boldsymbol{y} \in \left\{ \boldsymbol{y} | \boldsymbol{y} \in \mathbb{R}^{|G|}, ||\boldsymbol{y} - \boldsymbol{x}|| \leq \epsilon \right\}, \quad u(\boldsymbol{y}) > u(\boldsymbol{x}) \tag{25}$$

Convexity Preferences are convex if whenever $x \succcurlyeq y$ then

$$\lambda \boldsymbol{x} + (1 - \lambda) \boldsymbol{y} \succcurlyeq y \qquad \text{for all } \lambda \in [0, 1] \tag{26}$$

Convexity means that the agent prefers balanced bundles of goods to extreme bundles: if the agent is indifferent between $\boldsymbol{x}$ and $\boldsymbol{y}$ then they prefers the average $\lambda \boldsymbol{x} + (1 - \lambda) \boldsymbol{y}$ to either just $\boldsymbol{x}$ or $\boldsymbol{y}$. We can write this assumption in terms of utility functions. Preferences are convex if:

$$u(\lambda \boldsymbol{x} + (1 - \lambda) \boldsymbol{y}) \geq \min \left\{ u(\boldsymbol{x}), u(\boldsymbol{y}) \right\} \qquad \text{for all } \lambda \in [0, 1] \tag{27}$$

Slightly confusingly, a utility function that satisfies (27) is called **quasi-concave**. Note that any concave function is also quasi-concave.

Concavity Preferences are concave if whenever $y \succcurlyeq x$ then

$$\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} \preccurlyeq \boldsymbol{y} \quad \text{for all } \lambda \in [0,1] \tag{28}$$

Concavity means that the agent prefers extreme bundles of goods to balanced bundles: if the agent is indifferent between x and y then they prefers either just $\boldsymbol{x}$ or $\boldsymbol{y}$ to the average $\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}$. We can write this assumption in terms of utility functions. Preferences are convex if:

$$u(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \leq \max\{u(\boldsymbol{x}), u(\boldsymbol{y})\} \quad \text{for all } \lambda \in [0,1] \tag{29}$$

Slightly confusingly, a utility function that satisfies (29) is called **quasi-convex**. Note that any convex function is also quasi-convex.

2.3 Important Classes of Utility Functions

Let n be the number of goods. Let $\boldsymbol{x} \in \mathbb{R}^m$ be the vector representing the bundle of goods of the agent where the i^{th} entry represents the amount of good i. Let u be the utility function of the agent and let $\boldsymbol{V} \in \mathbb{R}^m$ be a vector of preference parameters for each good $i \in [m]$. We define a few important classes of utility functions.

Linear Utilities: This is the situation in which goods are perfect substitutes. That is, having more of a good leads the consumer to want less of other goods. An example of this is sugar and artificial sweeteners. If a buyer buys sugar, then it will not use artificial sweeteners. Mathematically, linear utilities are defined as:

$$u(\boldsymbol{x}) = \sum_{i=1}^{n} v_i x_i \tag{30}$$

Leontief Utilities: This is the situation in which goods are complements. That is in order to derive utility from one good, the buyer also needs to have more of other goods. An example is that a buyer needs both the left and right pair of a shoe to derive utility from the shoes. It cannot derive any utility from only the left or right shoe. Mathematically, Leontief utilities are defined as:

$$u(\boldsymbol{x}) = \min_{i=1,\ldots,n} \{v_i x_i\} \tag{31}$$

Cobb-Douglas Utilities: This is the in-between case between perfect substitutes and perfect complements. That is, a buyer prefers a bundle of goods that is balanced to bundles of goods that have more of a few goods. Mathematically, Cobb-Douglas utilities are defined as:

$$u(\boldsymbol{x}) = \prod_{i=1}^{n} x_i^{v_i} \tag{32}$$

where $\sum_{j=1}^{m} v_i = 1$.

These three utility functions happen to be special cases of the **constant elasticity of substitution (CES) utility** function which is defined as:

$$u(\boldsymbol{x}) = \left(\sum_{i=1}^{n} v_i x_i^{\rho}\right)^{\frac{1}{\rho}} \tag{33}$$

where $-\infty \leq \rho \leq 1$.

1. When $\rho = 1$ then the CES utility function is exactly the linear utility function
2. When $\rho \to -\infty$, the CES utility function is exactly the Leontief utility function.
3. When $\rho \to 0$, the CES utility function is exactly the Cobb-Douglas utility function

To see the above for the limits, one can use l'hopital's rule to calculate the limit of $\log(u(\boldsymbol{x}))$ at the desired limit value and from there one can deduce the value of $u(\boldsymbol{x})$ at the desired limit.[5] For $0 < \rho \leq 1$, goods are weak gross substitutes, for $\rho = 1$, goods are perfect substitutes, and for $\rho < 0$, goods are complementary.

Note: CES Utilities are homogeneous of degree 1

Proof:

$$u(\lambda \boldsymbol{x}) = \left(\sum_{i=1}^{n} v_i \lambda^{\rho} x_i^{\rho}\right)^{\frac{1}{\rho}} = (\lambda^{\rho})^{\frac{1}{\rho}} \left(\sum_{i=1}^{n} v_i x_i^{\rho}\right)^{\frac{1}{\rho}} = \lambda \left(\sum_{i=1}^{n} v_i x_i^{\rho}\right)^{\frac{1}{\rho}} \tag{34}$$

Note: For $\rho < 1$, the CES utility function is **strictly** concave while it is concave for $\rho \leq 1$. To see this you can take the second derivative of the utility function

The **elasticity of substitution** of CES utility functions is defined as $\frac{1}{1-\rho}$. Note that CES utilities are quasi-concave.

Let b be the budget of an agent. The budget constrained utility maximizing bundle, $\boldsymbol{x}^*$ of the agent for CES utilities with $\rho < 1$ is given by (for $\rho = 1$ the utility maximizing demand of the buyer at given prices is not unique since linear utilities are not strongly concave) :

$$x_i^* = b \frac{v_i^{1-\frac{\rho}{\rho-1}} p_i^{\frac{\rho}{\rho-1}-1}}{\sum_{k=1}^{n} v_k^{1-\frac{\rho}{\rho-1}} p_k^{\frac{\rho}{\rho-1}}} \tag{35}$$

Quasilinear Utilities: A quasilinear utility is one that obey the following form:

$$u\left(\boldsymbol{x}\right) = \alpha x_1 + \theta\left(x_2, \ldots, x_n\right) \tag{36}$$

where θ is an arbitrary function and $\alpha \geq 0$. More than often, when talking about quasilinear utilities in the literature, we refer utility functions of the following form which conforms with the general quasilinear form:

$$u\left(\boldsymbol{x}\right) = \sum_{1=1}^{n} x_i \left(v_i - p_i\right) \tag{37}$$

where p_i is the price of the i^{th} good.

[5]More on the limit calculation here

3 Fisher Market Model

The Fisher market model is the first model of a market that proves that a competitive equilibrium exists in a market, that is, there exists prices and allocation of goods that maximize the utility of buyers and clears the market. In other words, the fisher market proves that in a market prices exist such that the supply is equal to demand.

3.1 Market Elements

The market consists of:

1. Finite set of n heterogeneous **buyers** B.
2. Finite set of m heterogeneous **good** types G.
3. Each buyer $i \in [n]$ is characterized by:
   (a) a budget $b_i \in \mathbb{R}$. We assume that budgets are normalized $\sum_{i=1}^{n} b_i = 1$
   (b) a utility function, $u_i : \mathbb{R}^m \to \mathbb{R}_+$, giving the utility that buyer i derives from each bundle of goods.
4. WLOG, we assume that there is only one unit of each good $j \in [m]$ and that each good is demanded by at least one person. The results/solutions we provide in this section can be applied to the more general settings in which the number of copies of each good is different than 1 and there are goods that are not demanded.

3.2 Outcome of the Market

An **allocation** $\boldsymbol{X}$ is a map from goods to buyers, represented as a matrix, s.t. $X_{ij} \geq 0$ denotes the amount of good $j \in [m]$ allocated to buyer $i \in [n]$. Goods are assigned **prices** $\boldsymbol{p} \in \mathbb{R}^m$. Note that prices are **anonymous**, in that all copies of good j are assigned the same price p_j.
An allocation is **feasible** iff

- no more than 1 unit of a good $j \in [m]$ is allocated in total, across all buyers, $\sum\limits_{i \in [n]} X_{ij} \leq 1$ and
- buyers do not spend more than their budget: i.e., $\boldsymbol{p} \cdot \boldsymbol{X}_i \leq b_i$, for all $i \in [n]$, where $\boldsymbol{X}_i$ denotes the row vector corresponding to the allocation of buyer i in the matrix $\boldsymbol{X}$.

An **outcome** is a pair $(\boldsymbol{X}, \boldsymbol{p})$ consisting of an allocation and prices respectively.

3.3 Equilibrium

An outcome $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ is **utility maximizing** iff no buyer would prefer a different feasible allocation of goods than theirs, at the outcome's prices, that is $\forall i \in [n], \boldsymbol{X}_i, \;\; u_i(\boldsymbol{X}_i^*) \geq u_i(\boldsymbol{X}_i)$.

For any outcome $(\boldsymbol{X}, \boldsymbol{p})$, we define the **demand** for good j as $\sum\limits_{i=1}^{n} X_{ij}$ and the **supply** of good j as 1 (since we assumed that there is 1 unit of each good).

An outcome $(\boldsymbol{X}, \boldsymbol{p})$ is **market clearing** iff the demand of each good is equal to its supply, i.e. $\sum\limits_{i=1}^{n} X_{ij} = 1$,

An outcome is an **equilibrium** iff it is feasible, utility maximizing and market clearing. If there are goods that buyers do not demand, then those goods must be priced at zero.

The existence of the equilibrium for any continuous utility functions can be shown through the use of Sperner's lemma[6]. Sperner's lemma is a combinatorial analog of Brouwer's fixed point theorem, which posits the existence of a fixed of a function in a very general setting. This proof however is non-constructive in that we cannot use it to compute an equilibrium outcome of the Fisher market. We will provide a convex program to compute an equilibrium for the Fisher Market for any continuous concave utility functions. This also is a constructive proof of the existence of an equilibrium for the Fisher Market for any continuous concave utility function since the convex program is guaranteed to have an optimal value.

4 Connecting the Pari-Mutuel Model and the Fisher Market

We will now assume that the utility functions of the buyers are linear to provide an interesting connection between the optimal outcome of the Pari-Mutuel model and the Fisher Market. Let each buyer have preferences over goods represented as a vector of values $\boldsymbol{v}_i \in \mathbb{R}^m$, Linear utilities are defined as:

$$\forall i \in [n], u_i(\boldsymbol{X}_i) = \sum_{jin[m]} v_{ij} X_{ij} \tag{38}$$

It turns out that any equilibrium allocation of the fisher market is captured by the solution to the Eisenberg-Gale program. The corespondance between the variables in the both models is given in the table below:

Pari-Mutuel	**Fisher**
b_i (bettor budget)	B_i (buyer budget)
ξ_{ij} (proportion of B_i's bet in π_j)	X_{ij} (allocation of good j to buyer i)
p_{ij} (subjective probability)	v_{ij} (valuation)
π_j (final track probability)	p_j (price)
β_{ij} (bet of B_i on H_j)	$X_{ij}p_j$ (s pending of buyer i on good j)

It also turns out that while the Eisenberg-Gale program captures the equilibrium allocations, the dual of the program captures the equilibrium prices:

Primal

$$\max_{\boldsymbol{X}} \sum_{i=1}^{n} b_i \log\left(\sum_{j=1}^{m} v_{ij} X_{ij}\right) \tag{39}$$

$$\forall i \in [n], \qquad \sum_{j=1}^{m} X_{ij} \leq 1 \tag{40}$$

$$\forall i \in [n], j \in [m] \qquad X_{ij} \geq 0 \tag{41}$$

[6]https://en.wikipedia.org/wiki/Fisher_market#cite_note-3

Dual

$$\min_{\boldsymbol{p},\boldsymbol{\beta}} \sum_{j=1}^{m} p_j - \sum_{i=1}^{n} b_i \log(\beta_i) \tag{42}$$

$$\forall i \in [n], j \in [m] \qquad p_j \geq v_{ij}\beta_i \tag{43}$$

Notice that we can find any equilibrium allocation without knowing the prices (i.e., they are not part of the primal). We now generalize the Eisenberg-Gale convex program further so that it can provide a solution for any continuous concave utility functions.

Theorem 4.1. *Let n be the number of buyers and m be the number of goods in a Fisher market. If the utility function u_i for any consumer i is homogeneous of degree k (i.e., $\forall \boldsymbol{x} \in \mathbb{R}^m \quad \lambda > 0, \quad u_i(\lambda \boldsymbol{x}) = \lambda u_i(\boldsymbol{x})$) and concave (i.e., $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m, \quad \lambda \in (0,1), \quad u_i(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \lambda u_i(\boldsymbol{x}) + (1-\lambda)\boldsymbol{y}$), then the following program computes equilibrium allocations:*

$$\max_{\boldsymbol{X}} \sum_{i=1}^{n} b_i \log\left(u_i\left(\boldsymbol{X}_i\right)\right) \tag{44}$$

$$\forall j \in [m], \qquad \sum_{i=1}^{n} X_{ij} \leq 1 \tag{45}$$

$$\forall i \in [n], j \in [m] \qquad X_{ij} \geq 0 \tag{46}$$

Proof:
Without loss of generality, assume that the degree of the utility function is $k = 1$. This does not lose the generality of our result since we can transform any homogeneous function $f(x)$ of degree k, to a homogeneous function of degree 1 using the monotonically non-decresing transformation $\sqrt[k]{f(x)}$. This allows us to conserve all properties of the utility function, more specificaly the preference relations between goods by theorem 2.2. Firstly, note that since the utilities are concave and the logarithm function is a concave function the objective function is also concave. Furthermore, as the constraints are all affine, the program we propose is feasible and bounded. We then write down the the Lagrangian L for this convex program, using slack variables $\forall j \in [m], \ p_j \geq 0$ and $\forall j \in [m], i \in [n], \quad \lambda_{ij}$:

$$L(\boldsymbol{X}, \boldsymbol{p}, \boldsymbol{\lambda}) = \sum_{i=1}^{n} -b_i \log(u_i\left(\boldsymbol{X}_i\right)) + \sum_{j=1}^{m} p_j \left(\sum_{i=1}^{n} X_{ij} - 1\right) + \sum_{j=1}^{m}\sum_{i=1}^{n} \lambda_{ij}\left(-X_{ij}\right) \tag{47}$$

Any optimal solution of a convex program is guaranteed to satisfy a series of conditions called the Karush–Kuhn–Tucker (KKT) conditions.[7] From the complementary slackness condition, we have the following two conditions:

$$p_j\left(X_{ij} - 1\right) = 0 \tag{48}$$

$$\lambda_{ij} = 0 \tag{49}$$

From condition (48), we can also deduce the slack variable p_j corresponds to the equilibrium prices for the fisher market since this condition can be interpreted as walras' law.

[7]More background on this can be found here

From the stationarity conditions, we get:

$$\nabla L = \frac{-b_i}{u_i(\boldsymbol{X}_i)} \frac{\partial u_i}{\partial X_{ij}} + p_j - \lambda_{ij} = 0 \tag{50}$$

We can make two observations. Firstly, notice, that (48) implies that if $p_j > 0$ then $\sum_{i=1}^{m} = 1$ which is exactly the market clearance condition (the contrapositive of this statement also guarantees that if a good has a supply greater than the demand then it has a price of 0).

Secondly, we will manipulate (50) to prove that consumers do not spend more than their budget:

$$p_j = \frac{b_i}{u_i(\boldsymbol{X}_i)} \frac{\partial u_i}{\partial X_{ij}} + \lambda_{ij} \tag{51}$$

If $X_{ij} > 0$, using (49) we know that $\lambda_{ij} = 0$ which gives us:

$$p_j = \frac{b_i}{u_i(\boldsymbol{X}_i)} \frac{\partial u_i}{\partial X_{ij}} \tag{52}$$

By multiplying both side by X_{ij}, we obtain:

$$p_j X_{ij} = \frac{b_i X_{ij}}{u_i(\boldsymbol{X}_i)} \frac{\partial u_i}{\partial X_{ij}} \tag{53}$$

Summing up both sides across all goods, we obtain:

$$\sum_{j=1}^{m} p_j X_{ij} = b_i \sum_{j=1}^{m} \frac{X_{ij}}{u_i(\boldsymbol{X}_i)} \frac{\partial u_i}{\partial X_{ij}} \tag{54}$$

Before completing the proof, we need to prove one more theorem.

Theorem 4.2. ***Euler's Theorem***
Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a homogeneous function of degree k that is continuous and differentiable on $\mathbb{R}^n_{>0}$, then the following holds:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(\boldsymbol{x}) x_i = k f(\boldsymbol{x}) \tag{55}$$

Proof: Assume that f is a homogeneous function of degree k. Let $\boldsymbol{x} \in \mathbb{R}^n_{>0}$. Define $g : (0, \infty) \to \mathbb{R}$ such that:

$$g(\lambda) = f(\lambda \boldsymbol{x}) - \lambda^k f(\boldsymbol{x}) \tag{56}$$

Due to f being homogeneous, this function has a value of 0 for its entire domain. This implies that its derivative is also 0 for its domain:

$$g'(\lambda) = 0 \tag{57}$$

Using the chain rule, we also know that the derivative of g can also be calculated as:

$$g'(\lambda) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(\boldsymbol{x}) x_i - k f(\boldsymbol{x}) \tag{58}$$

Using (57) and setting $\lambda = 0$, we then get:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(\boldsymbol{x}) x_i = k f(\boldsymbol{x}) \tag{59}$$

Going back to the proof of theorem 4.1, we have:

$$\sum_{j=1}^{m} p_j X_{ij} = b_i \sum_{j=1}^{m} \frac{X_{ij}}{u_i(\boldsymbol{X}_i)} \frac{\partial u_i}{\partial X_{ij}} \tag{60}$$

$$\sum_{j=1}^{m} p_j X_{ij} = \frac{b_i}{u_i(\boldsymbol{X}_i)} \sum_{j=1}^{m} \frac{\partial}{\partial X_{ij}} u_i(\boldsymbol{X}_i) X_{ij} \tag{61}$$

$$\tag{62}$$

Using Euler's theorem with $k = 1$ (since by our assumption the utility functions are homogeneous of degree 1) we get:

$$\sum_{j=1}^{m} p_j X_{ij} = \frac{b_i}{u_i(\boldsymbol{X}_i)} \sum_{j=1}^{m} \frac{\partial}{\partial X_{ij}} u_i(\boldsymbol{X}_i) X_{ij} \tag{63}$$

$$\sum_{j=1}^{m} p_j X_{ij} = \frac{b_i}{u_i(\boldsymbol{X}_i)} u_i(\boldsymbol{X}_i) \tag{64}$$

$$\sum_{j=1}^{m} p_j X_{ij} = b_i \tag{65}$$

Notice that the left hand side of this expression is exactly the spending of any buyer. This result implies that consumers are not spending more than their budget.

We will now show that the optimal allocation given by this program is utility maximizing. Consider the objective function:

$$\max_{\boldsymbol{X}} \sum_{i=1}^{n} b_i \log\left(u_i\left(\boldsymbol{X}_i\right)\right) \tag{66}$$

We exponentiate this expression which preserve the optimal solution since exponentiation is a monotonically non-decreasing operation:

$$\max_{\boldsymbol{X}} \exp\left\{\sum_{i=1}^{n} b_i \log\left(u_i\left(\boldsymbol{X}_i\right)\right)\right\} \tag{67}$$

$$\max_{\boldsymbol{X}} \prod_{i=1}^{n} \exp\left\{b_i \log\left(u_i\left(\boldsymbol{X}_i\right)\right)\right\} \tag{68}$$

$$\max_{\boldsymbol{X}} \prod_{i=1}^{n} \exp\left\{\log\left(u_i\left(\boldsymbol{X}_i\right)^{b_i}\right)\right\} \tag{69}$$

$$\max_{\boldsymbol{X}} \prod_{i=1}^{n} u_i\left(\boldsymbol{X}_i\right)^{b_i} \tag{70}$$

$$\tag{71}$$

Notice that maximizing a product is equivalent to maximizing the individual elements of a product. Hence, the utilities of individual buyers are maximized (constrained by the budget - as proved previously). This concludes the proof as we have show that allocations given by the program we have given are feasible, utility maximizing and market clearing which means that they are equilibrium allocations. Finally, note that if $\forall i, j, \quad X_{ij} > 0$ then equilibrium prices can be computed as:

$$p_j = \frac{b_i}{u_i(\boldsymbol{X}_i)} \frac{\partial u_i}{\partial X_{ij}} \tag{72}$$

Furthermore, we can also find the dual of the program we proposed which gives us the equilibrium prices:

Dual

$$\min_{\boldsymbol{p},\boldsymbol{\beta}} \sum_{j=1}^{m} p_j - \sum_{i=1}^{n} b_i \log(\beta_i) \tag{73}$$

$$\forall i \in [n], j \in [m] \qquad p_j \geq \frac{\partial}{\partial X_{ij}} u_i(X_{ij}) \beta_i \tag{74}$$

One thing to note is that while quasi-linear utilities are concave and continuous, we cannot use the above convex program to find an equilibrium outcome for the quasi-linear case of the Fisher Market. This is because quasi-linear utilities include the prices which then cause the primal of the above program to no more be convex. More recent work has shown that another convex program, namely Shmyrev's convex program can be used to calculate the equilibrium outcome for a fisher market with quasi-linear utilities. The primal of the following program captures the equilibrium allocations while the dual captures the equilibrium prices:

Primal

$$\max_{\boldsymbol{X},\boldsymbol{u},\boldsymbol{v}} \sum_{i=1}^{n} b_i \log(u_i) - v_i \tag{75}$$

$$\forall i \in [n], \qquad u_i \leq \sum_{j \in [m]} v_{ij} X_{ij} + v_i \tag{76}$$

$$\forall i \in [n], \qquad \sum_{j \in [m]} X_{ij} \leq 1 \tag{77}$$

$$\forall i \in [n], j \in [m] \qquad X_{ij}, v_i \geq 0 \tag{78}$$

Dual

$$\min_{\boldsymbol{p},\boldsymbol{\beta}} \sum_{j=1}^{m} p_j - \sum_{i=1}^{n} b_i \log(\beta_i) \tag{79}$$

$$\forall i \in [n], j \in [m] \qquad p_j \geq v_{ij}\beta_i \tag{80}$$

$$\forall i \in [n], \qquad \beta_i \leq 1 \tag{81}$$

Note that for Cobb-Douglas utilities the equilibrium outcome $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ of the fisher market [1]:

$$X^*_{ij} = \frac{v_{ij}b_i}{\sum_{k\in[n]} v_{kj}b_k} \tag{82}$$

$$p^*_j = \sum_{i\in[n]} v_{ij}b_i \tag{83}$$

Proof: We assume that the valuations of the buyers are normalized such that $\sum_{j\in[m]} v_{ij} = 1$ Recall that the demand of buyer i for good j, $D^i(\boldsymbol{p})$ at given prices $\boldsymbol{p}$ for Cobb-Douglas utilities is:

$$D^i_j(\boldsymbol{p}) = \frac{v_{ij}b_i}{p_j} \tag{84}$$

Summing up both sides across all buyer, we get:

$$\sum_{i\in[n]} D^i_j(\boldsymbol{p}) = \sum_{i\in[n]} \frac{v_{ij}b_i}{p_j} \tag{85}$$

$$\sum_{i\in[n]} D^i_j(\boldsymbol{p}) = \frac{1}{p_j} \sum_{i\in[n]} v_{ij}b_i \tag{86}$$

The left hand side of this expression is the demand for good j. Since in a fisher market there is one unit of each good, we can set the demand equal to 1 and solve for the equilibrium price of good j, p^*_j:

$$1 = \frac{1}{p^*_j} \sum_{i\in[n]} v_{ij}b_i \tag{87}$$

$$p^*_j = \sum_{i\in[n]} v_{ij}b_i \tag{88}$$

$$\tag{89}$$

Finally, by substituting our formula for the equilibrium price of good j into the demand set formula, we get the equilibrium allocation X^*_{ij} of buyer i for good j:

$$X^*_{ij} = \frac{v_{ij}b_i}{\sum_{k\in[n]} v_{kj}b_k} \tag{90}$$

Note that these closed form solutions assumes that the valuations are of the following form, $\forall i \in [n],\ \sum_{j\in[m]} v_{ij} = 1$ and that there is only one unit of each good in the market.

5 Arrow-Debreu

The general equilibrium model of a competitive economy, also known as the Arrow-Debreu model, establishes the existence of a general equilibrium, that is prices, consumption and production that maximize firms profits, consumers' utilities and clears the market (1954).

5.1 Model Elements

The model consists of:

1. Finite set of l commodities (this can include raw goods, intermediate goods and labor)
2. Finite set of n production units (i.e., firms). Each firm $j \in [n]$ has:
   - a set of possible productions Y_j. An element $\boldsymbol{y}_j \in Y_j$ is a vector in $\mathbb{R}^l$. Positive elements of this vector are outputs while negative elements are inputs.
3. Finite set of m consumption units (i.e., agents/consumers). Every agent $i \in [m]$ has:
   - a set of possible consumptions X_i. An element $\boldsymbol{x}_i \in X_j$ is a vector in $\mathbb{R}^l$. Positive elements of this vector are commodities consumed while negative elements are the labor service that a consumer provides. We asumme that the labor that a consumer can provide is upperbounded.
   - a contractual claim to a share of the profits of each firm $\boldsymbol{\alpha}_i = (\alpha_{i1}, \dots, \alpha_{in})$
   - an endowment of commodities $\boldsymbol{e}_i = (e_{i1}, \dots, e_{il})$
   - prefereneces over goods $\boldsymbol{v}_i = (v_{i1}, \dots, v_{il})$
   - a utility function $u_i : \mathbb{R}^l \to \mathbb{R}$ parametrized by $\boldsymbol{v}_i$ that gives that the utility that an agent derives from a bundle of commodities.

5.2 Model Outcome

The price space is $P = \left\{\boldsymbol{p} \mid \ p_h \geq 0, \ \sum_{h=1}^{l} p_h = 1\right\}$. Commodities are assigned **prices** $\boldsymbol{p} \in P$. An outcome of the model is a tuple $(\boldsymbol{y}_1, \dots, \boldsymbol{y}_n, \boldsymbol{x}_1, \dots, \boldsymbol{x}_m, \boldsymbol{p}) \in Y_1 \times \cdots \times Y_n \times X_1 \times \cdots \times X_m \times P$.

An outcome $(\boldsymbol{y}_1, \dots, \boldsymbol{y}_n, \boldsymbol{x}_1, \dots, \boldsymbol{x}_m, \boldsymbol{p}) \in Y_1 \times \cdots \times Y_n \times X_1 \times \cdots \times X_m \times P$ is feasible iff the value of the consumption of agents is less than or equal to their income, i.e., $\forall i \in [m], \ \ \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p} + \sum_{j=1}^{n} \alpha_{ij} \boldsymbol{y}_j \cdot \boldsymbol{p}$.

5.3 Equilibrium

An outcome $(\boldsymbol{y}_1^*, \dots, \boldsymbol{y}_n^*, \boldsymbol{x}_1^*, \dots, \boldsymbol{x}_m^*, \boldsymbol{p}^*) \in Y_1 \times \cdots \times Y_n \times X_1 \times \cdots \times X_m \times P$ is an equilibrium iff:

1. Firms maximizes profit:
   $\forall j \in [n], \quad \boldsymbol{y}_j^*$ maximizes $\boldsymbol{p}^* \cdot \boldsymbol{y}_j$ over Y_j
2. Consumers maximize utility:
   $\forall i \in [m] \quad \boldsymbol{x}_i^*$ maximizes $u_i(\boldsymbol{x}_i; \boldsymbol{v}_i)$ over the set $\left\{\boldsymbol{x}_i \mid \ \boldsymbol{x}_i \in X_i \ \ \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^* + \sum_{j=1}^{n} \alpha_{ij} \boldsymbol{y}_j^* \cdot \boldsymbol{p}^*\right\}$

3. The markets clear and goods that are not demanded are priced at 0:
$\sum_{i=1}^{m} \boldsymbol{x}_i^* - \sum_{j=1}^{[n]} \boldsymbol{y}_j^* - \sum_{i=1}^{m} \boldsymbol{e}_i \leq 0$, and $\boldsymbol{p}^* \cdot \left(\sum_{i=1}^{m} \boldsymbol{x}_i^* - \sum_{j=1}^{[n]} \boldsymbol{y}_j^* - \sum_{i=1}^{m} \boldsymbol{e}_i \right) = 0$

Note that for the last condition, we need both mathematical statements since the first condition coupled with the second one ensures that if a good is under-demanded that it is priced at 0. (if it confuses you, the second part is more of a technical statement in a way because we assume that the prices cannot be negative) We now present the Arrow-Debreu Theorem due to Nobel prize laureate economists Kenneth J. Arrow and Gerard Debreu.

Theorem 5.1. ***The Arrow-Debreu Theorem I***
Suppose that the following conditions are satisfied:

1. *X_i is closed and convex for all $i \in [m]$*

2. *Y_j is closed and convex for all $j \in [n]$*

3. *All agents have a consumption that is stricly less than their endowment, i.e., for all agents $i \in [m]$, $\exists \boldsymbol{x}_i \in X_i, \quad \boldsymbol{x}_i < \boldsymbol{e}_i$*

4. *u_i is continuous*

5. *u_i is quasi-concave, i.e., $\forall \lambda \in (0,1), \ \ u_i(\lambda x + (1-\lambda) y) \geq \min \{u_i(x), u_i(y)\}$,*

6. *u_i is non-satiated, i.e., $\forall \boldsymbol{x} \in \mathbb{R}^l, \ \ \epsilon > 0, \ \ \exists \boldsymbol{y} \in \left\{ \boldsymbol{y} | \boldsymbol{y} \in \mathbb{R}^l, ||\boldsymbol{y} - \boldsymbol{x}|| \leq \epsilon \right\}, \quad u_i(\boldsymbol{y}) > u_i(\boldsymbol{x})$*

Then an equilibrium outcome of the Arrow-Debreu Model exists

Proof Sketch: We provide a sketch of the proof as it is a very involved proof.

Define the following optimization program whose output is the utility maximizing and budget constrained consumption bundle of consumer i (i.e., the demand of consumer i):

$$D^i(\boldsymbol{p}) = \left\{ \begin{array}{c} \arg\max_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i) \\ \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p} + \sum_{j=1}^{n} \alpha_{ij} \boldsymbol{y}_j \cdot \boldsymbol{p} \\ \boldsymbol{x}_i \in X_i \end{array} \right\} \tag{91}$$

where $\boldsymbol{y}_j$ is chosen arbitrarily.

By the assumptions of the theorem u_i is continuous, quasi-concave, and non-satiated and X_i is compact (because X_i is a closed subspace of $\mathbb{R}^l$), this means that the output of this program is unique for any input price vector $\boldsymbol{p}$ (meaning that it can be considered as regular function). Furthermore, these assumptions allow us to use a theorem called the maximum theorem that tells us that this function is continuous in its arguments (i.e., prices).

Define the following optimization program whose output is the profit maximizing production of firm j:

$$S^j(\boldsymbol{p}) = \left\{ \begin{array}{c} \arg\max_{\boldsymbol{y}_j} \boldsymbol{y}_j \cdot \boldsymbol{p} \\ \boldsymbol{y}_j \in Y_j \end{array} \right\} \tag{92}$$

By the assumptions of the theorem Y_j is compact (because $Y_{[n]}$ is a closed subspace of $\mathbb{R}^l$), since Y_j is convex, then the output of this program is unique (meaning that it can be treated like a regular function). Furthermore, by the maximum theorem, this means that this function is continuous in its arguments (i.e., prices).

Define the excess demand function for the economy that returns a vector of the differences in the supply and demand of each commodity:

$$Z(p) = \sum_{i\in[m]} D^i(\boldsymbol{p}) - \sum_{j\in[n]} S^j(\boldsymbol{p}) - \sum_{i\in[m]} \boldsymbol{e}_i \tag{93}$$

$$Z : P \rightarrow \mathbf{R}^N \tag{94}$$

Now define the function $T : P \rightarrow P$, that mimics a fictional auctioneer trying to bring the economy into an equilibrium by adjusting the prices based on the excess demand. More specifically this function is calculated as:

$$T_h(\boldsymbol{p}) = \frac{\max\left[0, p_h + \gamma_k Z_k(\boldsymbol{p})\right]}{\sum_{k\in[l]} \max\left[0, p_k + \gamma_k Z_k(\boldsymbol{p})\right]} \tag{95}$$

where is an arbitrary constant $\gamma_k > 0$.

The function T is a price adjustment function. It raises the relative price of goods in excess demand and reduces the price of goods in excess supply while keeping the price vector on the simplex. This function is also continuous since the demand and supply functions are continuous which implies that the excess demand function is continuous and since the $\max$ operator preserves continuity, T must also be continuous. Since T is continuous and maps from a convex compact set back to itself, by Brouwer's fixed point theorem, it has a fixed point. By the definition of the function this fixed point must be an equilibrium.[8]

Assumption 3 that every consumer has an endowment such that they could consume it and have left over endowment to sell is a very strong one. As a result Arrow and Debreu provide a second theorem with additional assumptions that gets rid of that assumption.

Theorem 5.2. ***The Arrow-Debreu Theorem II***
Suppose that the following conditions are satisfied:

1. *X_i is closed and convex for all $i \in [m]$*
2. *Y_j is closed and convex for all $j \in [n]$*
3. *Each agent has at least one good that they are endowed with, for which they have a consumption that does not consume that good entirely, i.e., $\forall i \in [m], \exists h \in [l], \quad x_{ih} < e_{ih}$.*
4. *There exists a consumption for all agents such the supply of goods is strictly greater than demand. More formally, Let $X = \left\{\boldsymbol{X} | \boldsymbol{X} = \sum_{i\in i} \boldsymbol{x}_i,\ where\ \boldsymbol{x}_i \in X_i\right\}$ and $Y = \left\{\boldsymbol{y} | \boldsymbol{y} = \sum_{j\in[n]} \boldsymbol{y}_j,\ where\ \boldsymbol{y}_j \in Y_j\right\}$ $\exists \boldsymbol{X} \in X, \boldsymbol{y} \in Y$, then $\exists \boldsymbol{X} \in X, \boldsymbol{y} \in Y, \quad \boldsymbol{X} < \boldsymbol{y} + \sum_{i\in[m]} \boldsymbol{e}_i$*
5. *u_i is continuous*
6. *u_i is quasi-concave, i.e., $\forall \lambda \in (0, 1),\ \ u_i(\lambda x + (1-\lambda)y) \geq \min\left\{u_i(x), u_i(y)\right\}$,*
7. *u_i is non-satiated, i.e., $\forall \boldsymbol{x} \in \mathbb{R}^l,\ \ \epsilon > 0,\ \ \exists \boldsymbol{y} \in \left\{\boldsymbol{y} | \boldsymbol{y} \in \mathbb{R}^l, ||\boldsymbol{y} - \boldsymbol{x}|| \leq \epsilon\right\}, \quad u_i(\boldsymbol{y}) > u_i(\boldsymbol{x})$*

Then an equilibrium outcome of the Arrow-Debreu Model exists

[8] Some of the mathematical details were skipped for brevity, you can find more detail notes here

6 Arrow-Debreu - Exchange Economy

The exchange economy is an important special case of an Arrow-Debreu economy in which there are no firms. That is, in this model, there is no production, rather we have traders who all would like to trade their endowments to improve their utilities. In the following sections, we define this model more formally.

6.1 Model Elements

1. Finite set of l commodities.

2. Finite set of m traders (i.e., agents/consumers). Every agent $i \in [m]$ has:

   - a set of possible consumptions X_i. An element $\boldsymbol{x}_i \in X_j$ is a vector in $\mathbb{R}^l_+$.
   - an endowment of commodities $\boldsymbol{e}_i = (e_{i1}, \dots, e_{il})$
   - preferences over goods $\boldsymbol{v}_i = (v_{i1}, \dots, v_{il})$
   - a utility function $u_i : \mathbb{R}^l \to \mathbb{R}$ parametrized by $\boldsymbol{v}_i$ that gives that the utility that an agent derives from a bundle of commodities.

6.2 Model Outcome

The price space is $P = \left\{\boldsymbol{p} \mid \ p_h \geq 0, \ \sum_{h=1}^{l} p_h = 1\right\}$. Commodities are assigned **prices** $\boldsymbol{p} \in P$. An outcome of the model is a tuple $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_m, \boldsymbol{p}) \in X_1 \times \cdots \times X_m \times P$.

An outcome $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_m, \boldsymbol{p}) \in X_1 \times \cdots \times X_m \times P$ is feasible iff the value of the consumption of agents is less than or equal to the value of their endowment, i.e., $\forall i \in [m], \ \ \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p}$.

6.3 Equilibrium

An outcome $(\boldsymbol{x}^*_1, \dots, \boldsymbol{x}^*_m, \boldsymbol{p}^*) \in X_1 \times \cdots \times X_m \times P$ is an equilibrium iff:

1. Traders maximize utility:
   $\forall i \in [m] \ \ \ \boldsymbol{x}^*_i$ maximizes $u_i(\boldsymbol{x}_i; \boldsymbol{v}_i)$ over the set $\{\boldsymbol{x}_i \mid \ \boldsymbol{x}_i \in X_i \ \ \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^*\}$

2. The markets clear and goods that are not demanded are priced at 0:
   $\sum_{i=1}^{m} \boldsymbol{x}^*_i \leq \sum_{i=1}^{m} \boldsymbol{e}_i$ and $\boldsymbol{p} \cdot (\sum_{i=1}^{m} \boldsymbol{x}^*_i - \sum_{i=1}^{m} \boldsymbol{e}_i) = 0$

Theorem 6.1. ***Arrow-Debreu (Exchange) Theorem***
Suppose that the following conditions are satisfied:

1. *X_i is closed and convex for all $i \in [m]$*

2. *Each agent has a consumption that will not consume its entire endowment, i.e., $e_{ih} > x_{ih}$ for all $i \in [m]$ and $h \in [l]$*

3. *u_i is continuous*

4. *u_i is quasi-concave, i.e., $\forall \lambda \in (0,1), \ \ u_i(\lambda x + (1-\lambda)y) \geq \min\{u_i(x), u_i(y)\}$,*

5. u_i *is non-satiated, i.e.,* $\forall \boldsymbol{x} \in \mathbb{R}^l, \;\; \forall \epsilon > 0, \;\; \exists \boldsymbol{y} \in \left\{ \boldsymbol{y} | \boldsymbol{y} \in \mathbb{R}^l, ||\boldsymbol{y} - \boldsymbol{x}|| \leq \epsilon \right\}, \quad u_i(\boldsymbol{y}) > u_i(\boldsymbol{x})$

Then an equilibrium outcome of the Arrow-Debreu Model exists

We skip the proof as the Arrow-Debreu Exchange model is equivalent to the Arrow-Debreu Competitive Economy model and the result from the Arrow-Debreu theorem directly apply to the exchange model.

Theorem 6.2. ***The Arrow-Debreu Theorem II***
Suppose that the following conditions are satisfied:

1. X_i *is closed and convex for all* $i \in [m]$
2. *Each consumer is endowed with at least one good for which they have a consumption that will not consume their entire endowment of that good, i.e.,* $\forall i \in [m], \exists h \in [l], \quad x_{ih} < e_{ih}$.
3. *There exists a consumption for all consumers such that they can consume less than the total endowment. More formally, let* $X = \left\{ \boldsymbol{X} | \boldsymbol{X} = \sum_{i \in i} \boldsymbol{x}_i, \text{ where } \boldsymbol{x}_i \in X_i \right\}$*, then* $\exists \boldsymbol{X} \in X, \quad \boldsymbol{X} < \sum_{i \in [m]} \boldsymbol{e}_i$
4. u_i *is continuous*
5. u_i *is quasi-concave, i.e.,* $\forall \lambda \in (0,1), \;\; u_i(\lambda x + (1-\lambda) y) \geq \min \left\{ u_i(x), u_i(y) \right\}$,
6. u_i *is non-satiated, i.e.,* $\forall \boldsymbol{x} \in \mathbb{R}^l, \;\; \forall \epsilon > 0, \;\; \exists \boldsymbol{y} \in \left\{ \boldsymbol{y} | \boldsymbol{y} \in \mathbb{R}^l, ||\boldsymbol{y} - \boldsymbol{x}|| \leq \epsilon \right\}, \quad u_i(\boldsymbol{y}) > u_i(\boldsymbol{x})$

Then an equilibrium outcome of the Arrow-Debreu Model exists

6.4 Fisher Market and Arrow-Debreu Exchange Economy

Theorem 6.3. *The Arrow-Debreu Exchange model is a generalization of the Fisher Market.*

Proof:
We provide a reduction from an arbitrary instance of the Fisher Market to an instance of the Arrow-Debreu Exchange Market. Let $(n, m, \boldsymbol{b}, u)$ Amy: utility is a vector; should be bold be an instance of the Fisher Market (this is a tuple made up of the number of buyers, number of goods, budgets for buyers and utility functions of buyers respectively).

We build the following Arrow-Debreu exchange market with $n+1$ consumers and $m+1$ goods. More specifically, we add an $(m+1)^{th}$ commodity which is money, and a new $(n+1)^{th}$ artificial consumer which initially will have all m goods, and is interested only in money.
We set the initial endowments of the consumers in this construction of the Arrow-Debreu Exchange model as follows:

Consumer	Commodity 1	...	Commodity m	Commodity $m+1$
1	0	...	0	b_1
2	0	...	0	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	0	...	0	b_n
$n+1$	1	...	1	0

The consumption set of the consumers are set as follows:

$$\forall i \in [n+1], \qquad X_i = \left\{ \boldsymbol{x}_i \mid \forall h \in [m+1],\ 0 \le x_{ih} \le \sum_{i \in [n+1]} e_{ih} \right\} \tag{96}$$

The new utility functions $u^{'}$ of the consumers are set as follows:

- The first n consumers, derive the utility given by u from the first m commodities and a utility of 0 for the $(m+1)^{th}$ commodity (i.e., money), i.e., $\forall i \in [n], u_i^{'}(\boldsymbol{x}_i) = u_i(\boldsymbol{x}_{i1}, \ldots, \boldsymbol{x}_{im})$
- The $(n+1)^{th}$ consumer, derives no utility from the first m commodities and a utility of x_{n+1m+1} from the $(m+1)^{th}$ commodity (i.e., money) where x_{n+1m+1} is the amount of money that the artificial consumer is allocated, i.e., $u_{n+1}^{'}(\boldsymbol{x}_{n+1}) = \boldsymbol{x}_{i,m+1}$

The equilibrium prices for the first m goods divided by the equilibrium price of good $m+1$ in this Arrow-Debreu Exchange market correspond exactly to the equilibrium prices for the Fisher market and the allocations of the first m goods correspond exactly to the equilibrium allocations for the Fisher Market.

Proof: First, we will show that for the Arrow-Debreu market that we built satisfies conditions of Arrow-Debreu's second theorem.

The consumption set of the consumers contains all of its end points as it bounded below by 0 and above by the total endowment of agents and those end points are included in the set. As a result it is a closed set. Furtheremore, the set is convex since any convex combination of two arbitrary points in the set belongs to the set. This can be derived by picking two arbitrary points and noticing that any convex combination of those two points will always respect the inequality condition defining the set. The consumption set of all consumers includes the vector of zeros, i.e., consumers consuming nothing. Since every consumer is endowed with the $(m+1)^{th}$ good (i.e., money), condition 2 is also fulfilled. Furthermore, this implies that there exists a consumption vector for consumers as a whole that is the vector of zeros. Since the supply of goods is strictly positive in the entire economy, then condition 3 is also fulfilled. Assuming that the utility functions in the Fisher market were continuous, then the utility functions we build are also continuous. This confirms condition 4. Assuming that the utility functions in the Fisher market were quasi-concave, then the utility functions we build are also quasi-concave since our transformations of the utility functions are monotonous transformations. This confirms condition 5. Assuming that the utility function in the Fisher market were non-satiated then they necessarily also are non-satiated in the Arrow-Debreu market for the first n consumers. Furthermore, the way we built the utility function of consumer $n+1$, getting more of the $(m+1)^{th}$ good strictly increases his utility (i.e., his utility function is monotonic) which implies non-satiation. This confirms condition 6.

Then we show that the equilibrium of the Arrow-Debreu market that we built can be used to the equilibrium of the Fisher market.

We will show that the equilibrium outcome of the Arrow-Debreu Exchange market we built gives us the equilibrium outcome of the Fisher Market.

Budget constraint: Any equilibrium outcome $(\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_m^*, \boldsymbol{p}^*)$ of the Arrow-Debreu model satisfies the feasibility condition, i.e.:

$$\forall i \in [n+1], \qquad \boldsymbol{x}_i^* \cdot \boldsymbol{p}^* \le \boldsymbol{e}_i \cdot \boldsymbol{p}^* \tag{97}$$

For agents, $1, \ldots, n$, since they are only endowed with the $(m+1)^{th}$ good, this condition can be restated as:

$$\forall i \in [n], \qquad \boldsymbol{x}_i^* \cdot \boldsymbol{p}^* \leq b_i p_{m+1}^* \tag{98}$$

Now, if we divide both sides by p_{m+1}^*, we obtain:

$$\forall i \in [n], \qquad \frac{1}{p_{m+1}^*} \boldsymbol{x}_i^* \cdot \boldsymbol{p}^* \leq b_i \tag{99}$$

Let $\boldsymbol{p}^f$ be the equilibrium prices calculated using the equilibrium prices $\boldsymbol{p}^*$ of the Arrow-Debreu exchange model we built, that is, for all $j = 1, \ldots, m$, $p_j^f = \frac{p_j^*}{p_{m+1}^*}$. Then, the previous expression becomes:

$$\forall i \in [n], \qquad \boldsymbol{x}_i^* \cdot \boldsymbol{p}^f \leq b_i \tag{100}$$

This confirms that the way we set prices satisfies the budget constraint of consumers in the Fisher market.

Utility Maximization: Firstly, any Arrow-Debreu exchange equilibrium maximizes the utility of the traders. This means that the allocation of goods must also maximize the utility of the buyers in the Fisher market. This is because traders' utility function in the Arrow-Debreu market does not derive any utility for the $(m+1)^{th}$ item (i.e., money). As a result, we know that the allocation of goods in the Arrow-Debreu market maximizes utility based on the m goods. Since for the first m goods, the utility functions of the traders is the same in both the Fisher market and the Arrow-Debreu market, this implies that the first m elements in the equilibrium consumption of the Arrow-Debreu market ensure utility maximization in the Fisher Market too.

Market Clearance Any Arrow-Debreu Equilibrium, is market clearing. to show that the market clearance condition caries to the Fisher market by setting the equilibrium prices in the fisher market as $\forall i \in [n] \boldsymbol{p}^f = \frac{p_j^*}{p_{m+1}^*}$, we will show that the demand set of the buyers is the same for both the Fisher Market and the Arrow-Debreu Market. Let $D^i(\boldsymbol{p})$ be the demand set of buyer i in the fisher market, and let $\Delta^i(\boldsymbol{p})$ be the demand set of the buyer in the Arrow-Debreu market:

$$\forall i \in [n] \qquad \Delta^i(\boldsymbol{p}^*) = \operatorname*{arg\,max}_{\boldsymbol{x}_i : \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^*} u_i^{'}(\boldsymbol{x}_i) \tag{101}$$

$$\forall i \in [n] \qquad \Delta^i(\boldsymbol{p}^*) = \operatorname*{arg\,max}_{\boldsymbol{x}_i : \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq b_i p_{m+1}^*} u_i^{'}(\boldsymbol{x}_i) \tag{102}$$

$$\forall i \in [n] \qquad \Delta^i(\boldsymbol{p}^*) = \operatorname*{arg\,max}_{\boldsymbol{x}_i : \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq b_i p_{m+1}^*} u_i(\boldsymbol{x}_i) \tag{103}$$

$$\forall i \in [n] \qquad \Delta^i(\boldsymbol{p}^*) = \operatorname*{arg\,max}_{\boldsymbol{x}_i : \frac{1}{p_{m+1}^*} \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq \frac{b_i}{p_{m+1}^*} p_{m+1}^*} u_i(\boldsymbol{x}_i) \tag{104}$$

$$\forall i \in [n] \qquad \Delta^i(\boldsymbol{p}^*) = \operatorname*{arg\,max}_{\boldsymbol{x}_i : \boldsymbol{x}_i \cdot \boldsymbol{p}^f \leq b_i} u_i(\boldsymbol{x}_i) \tag{105}$$

$$\forall i \in [n] \qquad \Delta^i(\boldsymbol{p}^*) = D^i(\boldsymbol{p}^f) \tag{106}$$

$$\sum_{i \in [n]} \Delta^i(\boldsymbol{p}^*) = \sum_{i \in [n]} D^i(\boldsymbol{p}^f) \tag{107}$$

Hence, since the demand $\sum_{i \in [n]} \Delta^i(\boldsymbol{p}^*)$ ensured that the first m goods cleared in the Arrow-Debreu market (because buyer $n+1$ does not demand any of the first m goods), then $\boldsymbol{p}^f$ must also clear the fisher market.

This means that $\boldsymbol{p}^f$ is the vector of prices that satisfies all competitive equilibrium conditions. Hence, we have shown that we can convert any instance of a fisher market to an arrow-debreu exchange market whose equilibrium maps back to the equilibrium of the fisher market.

6.5 Computing Arrow-Debreu Exchange Equilibria

We will now discuss the computational aspects of the Arrow-Debreu model. In a first time, we will discuss solving for Arrow-Debreu equilibria for the case when utility functions of the consumers are Cobb-Douglas, we will then introduce the first and only “natural" process that is guaranteed to converge to equilibrium prices a for a large class of utility functions. Note that the difficulty of finding Arrow-Debreu equilibria is entirely dependent of the structures of the utility functions of consumers (we will discuss the computational complexity of the Arrow-Debreu model for different classes of utility functions in the next few sections).

6.5.1 Arrow-Debreu - Cobb-Douglas (Eaves 1985)

We now consider the case of the Arrow-Debreu exchange market for which the utility function of the consumers is Cobb-Douglas. For this specific case, Curtis Eaves provided a fast and interesting algorithm [1]. Cobb-Douglas utilities are defined as:

$$\forall i \in [m] \qquad u_i(\boldsymbol{x}_i) = \prod_{h \in [l]} x_{ih}^{v_{ih}} \tag{108}$$

where we assume that $\forall i \in [n], \ \sum_{h \in [l]} v_{ih} = 1$

Given prices $\boldsymbol{p}$, we can calculate the demand d_h^i of consumer i for commodity h as follows (the proof follows directly from the closed form formula Marshallian demand for Cobb-Douglas utilities):

$$d_h^i(\boldsymbol{p}) = \frac{v_{ih}(\boldsymbol{p}^T \boldsymbol{e}_i)}{p_h} \tag{109}$$

Using this closed formula, we can calculate the excess demand Z_h for commodity h at given prices $\boldsymbol{p}$ as follows:

$$\begin{aligned} Z_h &= \sum_{i \in [m]} d_h^i(\boldsymbol{p}) - \sum_{i \in [m]} e_{ih} && (110) \\ &= \sum_{i \in [m]} \frac{v_{ih}(\boldsymbol{p}^T \boldsymbol{e}_i)}{p_h} - \sum_{i \in [m]} e_{ih} && (111) \end{aligned}$$

Denoting the valuation matrix for all agents by $\boldsymbol{V}$ and the endowment matrix for all agent by $\boldsymbol{E}$, we write the excess demand function Z in vector notation to obtain:

$$Z(\boldsymbol{p}) = D(\boldsymbol{p})^{-1} \boldsymbol{V}^T \boldsymbol{E} \boldsymbol{p} - \boldsymbol{E} \boldsymbol{j}_m \tag{112}$$

where $D(\boldsymbol{p})$ denotes the matrix whose diagonal entries are the prices for commodities. Note that since $\boldsymbol{V} \boldsymbol{j}_l = \boldsymbol{j}_m$, we can re-write the right hand-side of this expression as follows:

$$Z(\boldsymbol{p}) = D(\boldsymbol{p})^{-1}\boldsymbol{V}^T\boldsymbol{E}\boldsymbol{p} - \boldsymbol{E}\boldsymbol{V}\boldsymbol{j}_l \tag{113}$$

Remember that the marshallian demand function calculates the utility maximizing demand of the consumer, hence in order to find equilibrium prices we just need to set the excess demand to obtain maket clearance and get prices for an Arrow-Debreu equilibrium. That is, we are looking for price that satisfy the following condition:

$$D(\boldsymbol{p})^{-1}\boldsymbol{V}^T\boldsymbol{E}\boldsymbol{p} - \boldsymbol{E}\boldsymbol{V}\boldsymbol{j}_l = 0 \tag{114}$$

Note that in this specific case Eaves uses a stricter definition of equilibrium where we need to have exact market clearance (i.e. excess demand is equal to 0) as opposed to the classical definition of market clearance provided by Arrow-Debreu, which allows 0 prices if excess demand is negative for any strictly positive price (i.e. the good is not demanded by any agent). That is, Eaves is looking for only strictly positive prices to the above equation. This is important since this means that equilibrium prices might not always exists (i.e., the Arrow-Debreu theorems' equilibrium existence proofs are not constructed for strictly positive prices).

Solving for prices for the above system of equations is equivalent to solving for prices in the following system:

$$(D(\boldsymbol{p})\boldsymbol{V}^T\boldsymbol{E})^T\boldsymbol{j}_l - (D(\boldsymbol{p})\boldsymbol{V}^T\boldsymbol{E})\boldsymbol{j}_l = 0 \tag{115}$$

This system is also equivalent to the following system:

$$(E - D(\boldsymbol{V}^T\boldsymbol{E}\boldsymbol{j}_l))^T\boldsymbol{p} = 0 \tag{116}$$

As previously mentioned, solving this equation alone is not enough since strictly positive prices might not exist. We need to also discuss existence of strictly positive prices. The above equation tells us that the existence (and uniqueness) of equilibrium prices is solely based on the matrix $\boldsymbol{V}^T\boldsymbol{E}$ and not only on its individual components.

Definition 6.4. *A matrix is* ***line-sum-symmetric*** *iff its corresponding row and column sums are equal.*

Namely, observing equation (115), we can see that in order for strictly positive equilbrium prices to exist, we need to prove the existence of prices $D(\boldsymbol{p})$ such that $D(\boldsymbol{p})\boldsymbol{V}^T\boldsymbol{E}$ is line-sum-symmetric. In the words of Eaves, "solving for prices is the task of finding a positive row-scaling $\boldsymbol{p}$, of $\boldsymbol{V}^T\boldsymbol{E}$ which is line-sum-symmetric.

In order to describe the necessary and sufficient conditions for the existence of strictly positive prices, we need to introduce one more concept.

Definition 6.5. *Given two goods i and j, good i is defined to* ***access*** *good j iff $i = j$ or if there is a sequence of goods $i = g_1, g_2, \dots, g_r, j = g_{r+1}$ such that $(\boldsymbol{V}^T\boldsymbol{E})_{g_k, g_{k+1}} > 0$ for $k = 1, \dots, r$.*

In other words, a good i accesses good j if there is a sequence of $t_1, \dots, t_r$ such that agent t_k possesses good k and desires good g_{k+1} for $k = 1, \dots, r$.

Definition 6.6. *The matrix $\boldsymbol{V}^T\boldsymbol{E}$ is defined to have symmetric access if for every pair of goods i and j they access each other or neither accesses the other.*

Definition 6.7. *The matrix $\boldsymbol{V}^T\boldsymbol{E}$ is defined to have full access iff for every pair of goods i and j, i and j access each other.*

Using all these definitions, we now present a theorem proven by Eaves in an earlier paper:

Theorem 6.8. *A square non-negative matrix $\boldsymbol{V}^T\boldsymbol{E}$ has a line-sum-symmetric positive row scaling iff $\boldsymbol{V}^T\boldsymbol{E}$ has symmetric access. A square non-negative matrix $\boldsymbol{V}^T\boldsymbol{E}$ has a line-sum-symmetric positive row scaling that is unique iff $\boldsymbol{V}^T\boldsymbol{E}$ has full access.*

This shows that strictly positive equilibrium prices exist iff $\boldsymbol{V}^T\boldsymbol{E}$ has symmetric access. This brings us to the next theorem which we do not prove as it is relatively complicated.

Theorem 6.9. *The Cobb-Douglas Arrow-Debreu exchange market $(\boldsymbol{V}, \boldsymbol{E})$ has unique strictly positive prices iff $\boldsymbol{V}^T\boldsymbol{E}$ has symmetric access.*

Before getting to the computation of the equilibrium we discuss the concept of submarkets.

Definition 6.10. *A **submarket** is a subset of agents and goods such that any agent possesses and desires only goods in the submarket and does not own or desire goods outside of the submarket.*

Note that every good in a submarket has full access.

It turns out that we can use a very fast decomposition algorithm (that runs in $O(m^2)$ time for a square matrix of size $m \times m$) on the matrix $\boldsymbol{V}^T\boldsymbol{E}$ such that we can obtain M independent submarkets $(\boldsymbol{B}_k, \boldsymbol{W}_k)$ for $k = 1, \ldots, M$. Since these markets are submarkets, for $k = 1, \ldots, M$, $\boldsymbol{B}_k^T\boldsymbol{W}_k$ has a unique row scaling. This means that we can use Gaussian elimination that runs in $O(m^3)$ time for a square matrix of size $m \times m$ to obtain the prices of the goods in each submarket and then combine the prices in each submarket and normalize them to obtain the equilbrium prices for the Arrow-Debreu market. This approach takes a running time of $O(m^3)$ for a matrix $\boldsymbol{V}^T\boldsymbol{E}$ of size $m \times m$.

6.5.2 The Tatonnement Process

The Tatonnement process (from French "Trial and error") is a process guaranteed to converge to equilibrium prices allocations for a class of utility functions called **Gross Substitutes**. The Tatonnement process (also called the Walrassian auction) was invented way before the Arrow-Debreu model. Briefly, it is an auction that adjusts the prices of the goods based on the excess demand of goods.

Define the following optimization program whose output is the utility maximizing and budget constrained consumption bundle of consumer i (i.e., the demand of consumer i):

$$D^i(\boldsymbol{p}) = \left\{ \begin{array}{c} \arg\max_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i) \\ \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p} + \sum_{j=1}^{n} \alpha_{ij}\boldsymbol{y}_j \cdot \boldsymbol{p} \\ \boldsymbol{x}_i \in X_i \end{array} \right\} \tag{117}$$

where $\boldsymbol{y}_j$ is chosen arbitrarily.

Define the excess demand function $Z : P \to P$:

$$Z(\boldsymbol{p}) = \sum_{i\in[m]} D^i(\boldsymbol{p}) - \sum_{i\in[m]} \boldsymbol{e}_i \tag{118}$$

We now define the Gross Substitutes condition within the context of the Arrow-Debreu model (which is different than the definition of the gross substitutes condition for auctions with indivisible items but these definitions are related). In the context of the Arrow-Debreu model, the Gross Substitute condition is a characteristic of the excess demand function rather than that of the utility functions.

Gross Substitutes: Let $\boldsymbol{p}$ and $\boldsymbol{p}^{'}$ bet two different price vectors such that $\forall h \in [l], p_h \leq p_h^{'}$ and $\exists k \in [l]$ such that $p_k < p_k^{'}$. Then an excess demand function Z fulfills the Gross Substitutes condition iff:

$$\forall h \neq k, \;\; Z_h(\boldsymbol{p}) > Z_k(\boldsymbol{p}^{'}) \tag{119}$$

where Z_h denotes the h^{th} coordinate in the output vector of the excess demand function (i.e., the excess demand for the commodity h) Equivalently in calculus terms, we can state the gross substitutes condition as:

$$\forall h \neq k, \;\; \frac{\partial Z_h}{\partial p_k} > 0 \tag{120}$$

In other words, if the prices of some goods are increased while the prices of some other goods are held fixed, this can only cause an increase in the demand of the goods whose price stayed fixed.

Before introducing the Tatonnement process we will introduce two more conditions that hold under the Arrow-Debreu Theorem Assumptions.

Homogeneity: $\forall \alpha > 0, Z(\alpha \boldsymbol{p}) = Z(\boldsymbol{p})$.
Proof:

We will first show that multiplying prices by a strictly positive scalar does not change demand.

$$\begin{aligned}
D^i(\alpha \boldsymbol{p}) &= \left\{ \begin{array}{c} \arg\max_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i) \\ \boldsymbol{x}_i \cdot \alpha \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \alpha \boldsymbol{p} \\ \boldsymbol{x}_i \in X_i \end{array} \right\} && (121) \\
&= \left\{ \begin{array}{c} \arg\max_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i) \\ \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p} \\ \boldsymbol{x}_i \in X_i \end{array} \right\} && (122) \\
&= D^i(\boldsymbol{p}) && (123)
\end{aligned}$$

$$\begin{aligned}
Z(\alpha \boldsymbol{p}) &= \sum_{i \in [m]} D^i(\alpha \boldsymbol{p}) - \sum_{i \in [m]} \boldsymbol{e}_i && (124) \\
&= \sum_{i \in [m]} D^i(\boldsymbol{p}) - \sum_{i \in [m]} \boldsymbol{e}_i && (125) \\
&= Z(\alpha \boldsymbol{p}) && (126)
\end{aligned}$$

Walras' Law: $\boldsymbol{p} \cdot Z(\boldsymbol{p}) = 0$, meaning that the total spending and total income in the economy are equal to each other.
Proof:

$$\begin{aligned}
\boldsymbol{p} \cdot Z(\boldsymbol{p}) &= \boldsymbol{p} \cdot \left(\sum_{i \in [m]} D^i(\boldsymbol{p}) - \sum_{i \in [m]} \boldsymbol{e}_i \right) &(127)\\
&= \boldsymbol{p} \cdot \sum_{i \in [m]} D^i(\boldsymbol{p}) - \boldsymbol{p} \cdot \sum_{i \in [m]} \boldsymbol{e}_i &(128)\\
& &(129)
\end{aligned}$$

Remember from the Arrow-Debreu Theorem conditions that the utility of the agents are non satiated and quasi-concave. An implication of this is that, in order for the agent to maximize their utilities, they have to spend their entire budget. That is, agent's spending is equal to the value of the bundle Mathematically, this gives:

$$\boldsymbol{p} \cdot \sum_{i \in [m]} D^i(\boldsymbol{p}) = \boldsymbol{p} \cdot \sum_{i \in [m]} \boldsymbol{e}_i \tag{130}$$

Hence, going back to the original problem we get:

$$\begin{aligned}
\boldsymbol{p} \cdot Z(\boldsymbol{p}) &= \boldsymbol{p} \cdot \sum_{i \in [m]} D^i(\boldsymbol{p}) - \boldsymbol{p} \cdot \sum_{i \in [m]} \boldsymbol{e}_i &(131)\\
&= \boldsymbol{p} \cdot \sum_{i \in [m]} \boldsymbol{e}_i - \boldsymbol{p} \cdot \sum_{i \in [m]} \boldsymbol{e}_i &(132)\\
&= 0 &(133)
\end{aligned}$$

We now introduce the Tatonnement process:

Tatonnement Process: Let $G(.)$ be a monotonous sign preserving function. The Tatonement is a time process, which at each time step t changes prices in the following manner:

$$\frac{dp_h}{dt} = G(Z_h(\boldsymbol{p})) \tag{134}$$

In words, the Tatonnement process, increases the prices of goods that are demanded in excess and decreases the prices of the goods are supplied in excess at each time step.

Theorem 6.11. *If the excess demand function satisfies the homogeneity, Walras' Law and Gross Substitutes conditions, then the Tatonnement process converges to the equilibrium prices* $\mathbf{p}^*$ *for the Arrow-Debreu Model.*

Note that these conditions are **necessary** for the Tatonnement process to converge to equilibrium prices but we will only prove that they are sufficient.

Proof:
To prove this result, we introduce a theorem proved by Kenneth Arrow and Leonid Hurwicz in the 60s [2]. As the proof is relatively involved, we skip it.

Theorem 6.12. ***Weak Axiom of Revealed Preferences***
If the excess demand Z satisfies the homogeneity, Walras' law, gross substitutes conditions, then for every non-equilibrium price $\boldsymbol{p}$ and equilibrium price $\boldsymbol{p}^*$, we have:

$$\sum_{h\in[l]} p_h^* Z_h(\boldsymbol{p}) > 0 \tag{135}$$

We will now establish the convergence of the Tatonnement Process using a Lyapunov potential function, a method used in Dynamic Systems to establish the convergence of time processes to equilibria. We define the following potential function:

$$V(\boldsymbol{p}) = \frac{1}{2}\sum_{h\in[l]} (p_h - p_h^*)^2 \tag{136}$$

The idea behind the potential function is to calculate the distance between the price vector calculated by the Tatonnement process at any time step and the equilibrium price vector. Then, if we can show that for each successive price vector obtained by the Tatonnement process this distance decreases we essentially have proven that asymptotically the process needs to output the equilibrium price vector. To do so, we now parametrize the price vector outputted by the tatonnement process at each time step t by t. We then get the following potential function:

$$V(\boldsymbol{p}) = \frac{1}{2}\sum_{h\in[l]} (p_h(t) - p_h^*)^2 \tag{137}$$

Taking the derivative of this potential function with respect to t, we get:

$$\frac{dV}{dt} = \sum_{h\in[l]} (p_h(t) - p_h^*)\,\frac{dp_h}{dt} \tag{138}$$

By the definition of the Tatonnement process we have:

$$\frac{dp_h}{dt} = G(Z_h(\boldsymbol{p})) \tag{139}$$

If we pick $G(\boldsymbol{p}) = \boldsymbol{p}$ (which is a monotonous sign preserving function) and we substitute it into the derivative of the potential function, we get:

$$\frac{dV}{dt} = \sum_{h\in[l]} (p_h(t) - p_h^*)\, Z_h(p(t)) \tag{140}$$

$$= \sum_{h\in[l]} p_h(t)Z_h(p(t)) - \sum_{h\in[l]} p_h^* Z_h(p(t)) \tag{141}$$

$$\tag{142}$$

By Walras' law, we know that $\sum_{h\in[l]} p_h(t)Z_h(p(t)) = 0$ and by the weak axiom of revealed preferences, we know that $p_h^* Z_h(p(t)) > 0$. We then get:

$$
\begin{aligned}
\frac{dV}{dt} &= \sum_{h \in [l]} p_h(t) Z_h(\boldsymbol{p}(t)) - \sum_{h \in [l]} p_h^* Z_h(\boldsymbol{p}(t)) && (143) \\
&= -\sum_{h \in [l]} p_h^* Z_h(\boldsymbol{p}(t)) && (144) \\
&< 0 && (145)
\end{aligned}
$$

The derivative of the potential function with respect to each time step t being negative implies that at each iteration of the Tatonnement process the distance between the price vector outputted by the Tatonnement process and the equilibrium price vector only decreases which means that as t goes to infinity the Tatonnement process's output price vector converges to the equilibrium price vector.

7 Computational Complexity of Finding Equilbria

7.1 Preliminaries

Decisions Problems Let $\Sigma := \{0, 1\}$. We can view a decision problem $D : \Sigma^* \to \{0, 1\}$ as the language

$$L = \{x \in \Sigma^* : D(x) = 1\} \tag{146}$$

that is given a problem instance $x \in \Sigma^*$ the decider D outputs a 1 (YES) or 0 (NO) to answer a **decision problem**. If the decision problem is "Given two numbers x and y, does x divide y evenly", then the decider checks whether if for given x and y, x divides y evenly (YES) or x does not divide y evenly (NO). Hence, the language corresponds to the set of all inputs that the decider answers YES to.

Polynomial Time Problems : A language L (i.e., a decision problem) is in the complexity class P (**deterministic polynomial time**) if there exists a **deterministic Turing machine** that decides in polynomial time whether or not $x \in \Sigma^*$ is in Z. We say that such a problem is **decidable in polynomial time**.

Non-deterministic Polynomial Time Problems : A language L (i.e., a decision problem) is in the complexity class NP (nondeterministic polynomial time) it has a polynomial time **verifier**. A verifier for a decision problem is an algorithm V that takes an an instance x of the decision problem , and a **certificate** c, some additional information C and output 1 (YES) or (NO). The certificate c provides the verifier with information that allows it to confirm or not whether if the the problem instance x is in the language L. If the decision problem is "Given two numbers x and y, does x divide y evenly", a certificate is a number c. Then the verifier is the algorithm that outputs 1 if $xc = y$ and c is an integer, otherwise it outputs 0.

More formally, L is in NP if and only if there exists a deterministic Turing machine V so that

$$L = \left\{x \in \Sigma^* : \text{ there exists a certificate } c \in \Sigma^* \text{ with } \forall c > 0, \ \ |y| = O\left(|x|^k\right) \text{ such that } V(x, c) = 1\right\}$$

We say that algorithm V verifies language Z in polynomial time. An equivalent definition is, a language L (i.e., a decision problem) is in the complexity class P (**deterministic polynomial time**) if there exists a **non-deterministic Turing machine** that decides in polynomial time whether or not $x \in \Sigma^*$ is in Z. We say that the problem is decidable in **Non-Deterministic Polynomial Time**.

Search Problems So far, we have discussed whether if a solution exists or not for a problem (i.e., a YES/NO answer). However, more often than not, we are concerned by finding the solution itself! **A search problem** Q is a relation on the set $R \subseteq \Sigma^* \times \Sigma^*$. [9] A pair $(I, s) \in R$ represent an instance I of the problem and a solution s of the problem.

7.2 PLS Complexity Class

The Polynomial Local Search class of problems

[9] Note that we are talking of a relation and not a function here since a particular instance of the problem $x \in \Sigma^*$ might not have any solutions or might have multiple ones.

7.3 PPAD Complexity Class

7.4 Complexity of Finding Fisher Market Equilibria

7.5 Complexity of Brouwer's Fixed Point

7.6 Complexity of Finding Arrow-Debreu Equilibria

7.6.1 Gross Substitutes

Theorem 7.1. *If the excess demand Z satisfies the homogeneity, Walras' law and the gross substitutes conditions, then we can find an equilibrium price vector in polynomial time.*

Proof:

The weak axiom of revealed preferences provides us with an "oracle" that allows us to seperate the space of potential solutions into two. Namely, let $S(\boldsymbol{p})$ be the set of possible solutions any current guess of prices $\boldsymbol{p}$. We have $S(\boldsymbol{p}) = \left\{\boldsymbol{p}^* | \sum_{h\in[l]} p_h^* Z_h(\boldsymbol{p}) > 0\right\}$. Since we have a separation oracle, we can then use the ellipsoid method to find an equilibrium price. What this method does is that it makes a guess, then calculates a new set of feasible solution using the separation oracle until it guesses a price vector for which $\sum_{h\in[l]} p_h^* Z_h(\boldsymbol{p}) = 0$. The ellipsoid method converges to the solution in polynomial time which means that we can find equilbirium prices in polynomial time.

7.6.2 Leontief Utilities

In this section we provide a reduction from a specific instance of an Arrow-Debreu Leontief to 2NASH which proves that the computation of Arrow-Debreu Leontief equilbria is PPAD-hard. We first describe the instance of Arrow-Debreu Leontief economies called "pairing economies" which reduce to the problem of solving non-zero sum 2 player nash equilibria. A pairing economy is a specific type of an Arrow-Debreu economy where:

1. there are $n + m$ traders
2. there are $n + m$ goods
3. each trader $i \in \{1, \ldots, n\}$ is endowed with a unique good $j \in \{1, \ldots, n\}$
4. each trader $i \in \{n + 1, \ldots, m\}$ is endowed wih unique good $j \in \{n + 1, \ldots, m\}$
5. each trader $i \in \{1, \ldots, n\}$ desires only goods $\{n + 1, \ldots, m\}$
6. each trader $i \in \{n + 1, \ldots, m\}$ desires only goods $\{1, \ldots, n\}$

Theorem 7.2. *Let (A, B) denote an arbitrary bimatrix game, where we assume, w.l.o.g., that the entries of the matrices A and B are all positive. Let the columns of*

$$H = \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix}$$

describe the utility parameters of the traders in a two groups Leontief economy. There is a one-to-one correspondence between the Nash equilibria of the game (A, B) and the market equilibria of the two-groups Leontief economy.

Furthermore, the correspondence has the property that a strategy is played with positive probability at a Nash equilibrium if and only if the good held by the corresponding trader has a positive price at the corresponding market equilibrium.

7.6.3 Summary of Complexity of Equilibria Computation

Problem	**Complexity**
Pure Nash Computation	PLS-Complete
Mixed Nash for #Players > 2	PPAD-Complete
Correlated Equilibria	Polynomial Time
Brouwer Fixed Point	PPAD-Complete

7.6.4 Summary of Complexity of Economic Equilibria Computation

8 Summary of Equilibrium Complexities

Problem	**Utilities**	**Complexity**
Fisher Market	Linear	Polynomial Time
Fisher Market	Cobb-Douglas	Polynomial Time
Fisher Market	Leontief	Polynomial Time
Fisher Market	Quasi-Linear	Polynomial Time
Fisher Market	CES with $-\infty < \rho \leq 1$	Polynomial Time
Fisher Market	Concave Homogenous Continuous	Polynomial Time
Fisher Market	Additively separable, piecewise-linear and concave	PPAD-hard
Arrow-Debreu	Linear	Strongly Polynomial Time
Arrow-Debreu	Gross Substitutes (e.g. CES with $\rho \geq -1$)	Polynomial Time
Arrow-Debreu	Quasilinear	Open
Arrow-Debreu	Leontief	PPAD-hard
Arrow-Debreu	CES with $-\infty < \rho < -1$	Open
Arrow-Debreu	Additively separable, piecewise-linear and concave	PPAD-Complete / no PTAS

References

[1] B. Curtis Eaves. *Finite solution of pure trade markets with Cobb-Douglas utilities*, pages 226–239. Springer Berlin Heidelberg, Berlin, Heidelberg, 1985.

[2] Kenneth J. Arrow and Leonid Hurwicz. On the stability of the competitive equilibrium, i. *Econometrica*, 26(4):522–552, 1958.